Quantum metrology in the finite-sample regime

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Quantum metrology in the finite-sample regime

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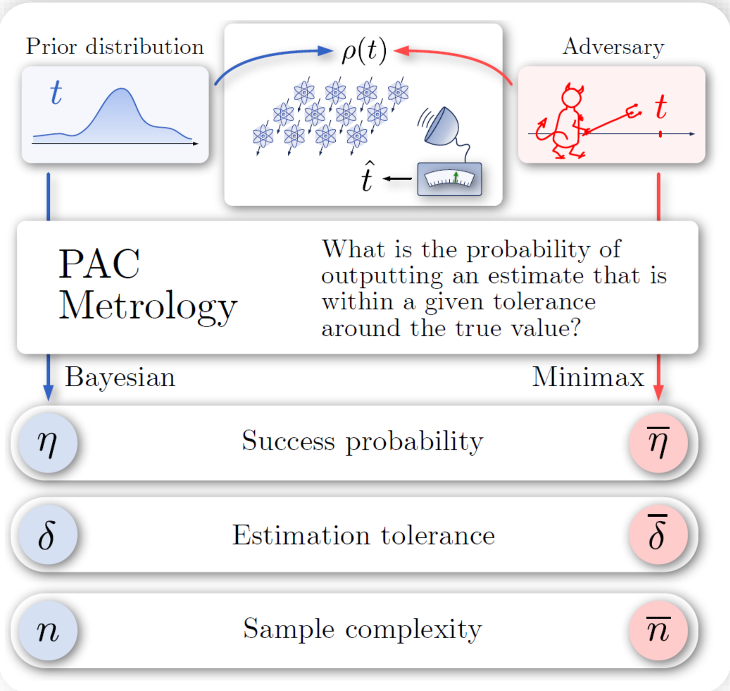
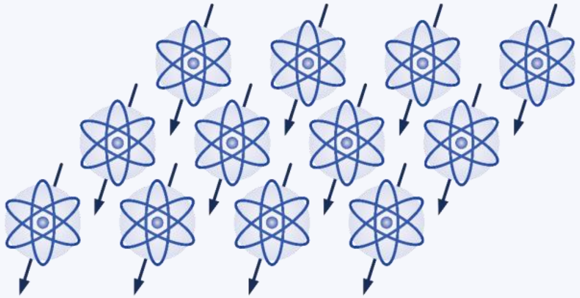
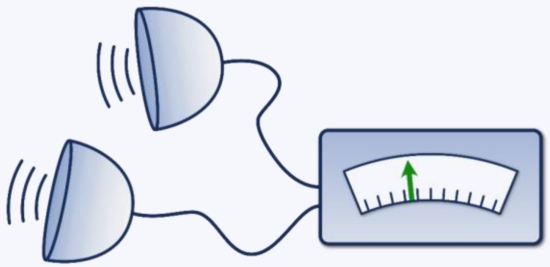





Traditional Quantum Metrology



$\rho(t)$



$Q(\hat{t})$

Want unbiased estimate

$$\mathbb{E}[\hat{t}] = t$$

with **low variance**

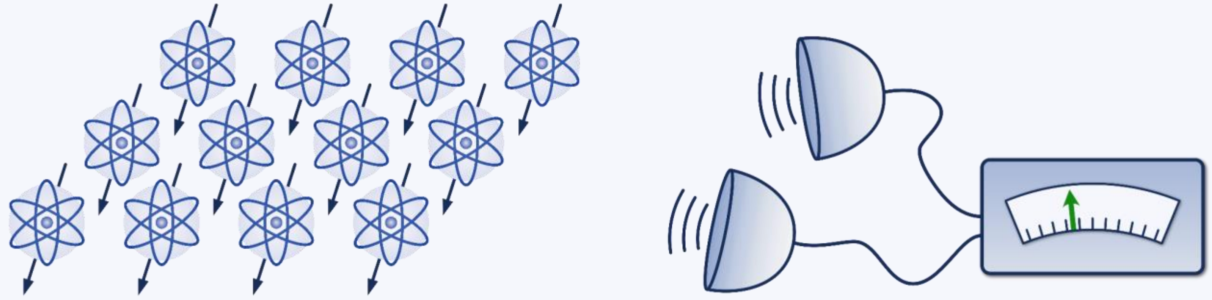
Cramér-Rao Bound

$$\mathrm{Var}(\hat{t}) \geq \frac{1}{\mathcal{F}(t)}$$

- Inherently asymptotic
- Assumes parameter is already approximately known
- Application difficult to justify in the finite-sample regime

Single-shot Quantum M

Estimation Tolerance



$\rho(t)$ $\qquad$ $Q(\hat{t})$

Is $|t - \hat{t}| \leq \boxed{\delta}$?

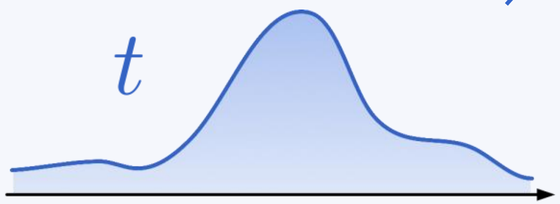
Yes $\longrightarrow$ **Success** ✓

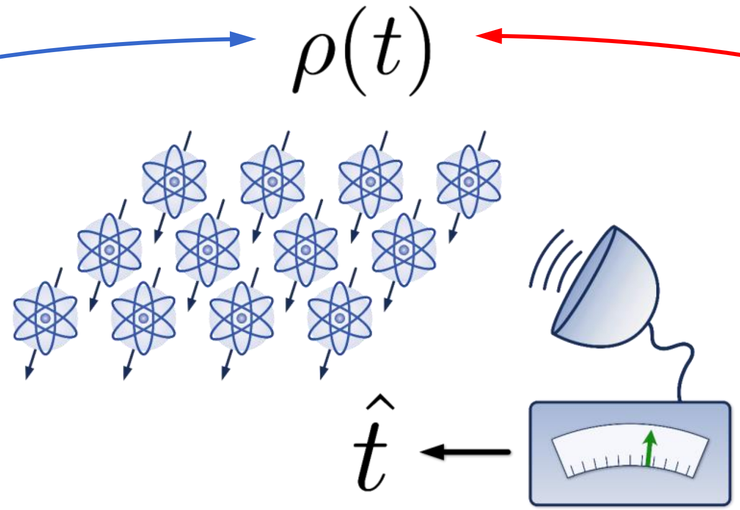
No $\longrightarrow$ **Failure** ✗

What is the probability of successful estimation?

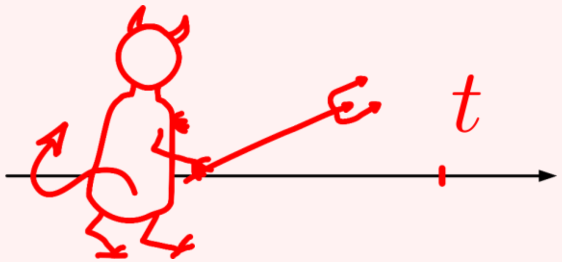
Single-shot Quantum Metrology

Prior distribution





Adversary



Bayesian success probability

$$\eta = \int d\mu(t) \int_{-\delta}^{\delta} d\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)]$$

Minimax success probability

$$\overline{\eta} = \min_{t} \int_{-\delta}^{\delta} d\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)]$$

Optimal Measurements

Optimal minimax success probability

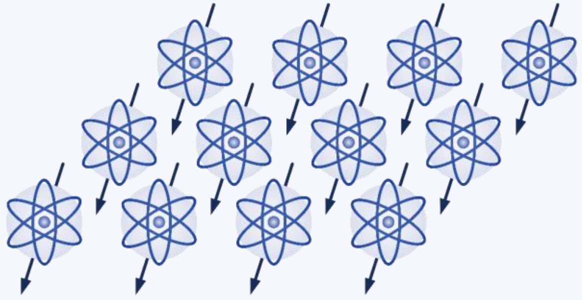
$$\overline{\eta}^* = \boxed{\max_{Q(\hat{t})}} \left\{ \min_t \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)] \right\}$$

Constitutes a **semi-infinite program**,
think a continuous semi-definite program

- We give a dual formulation without duality gap
- We generalize it to the parametrized channels where we optimize over combs or strategies with indefinite causal order
- We also give post-processing strategies for fixed measurements

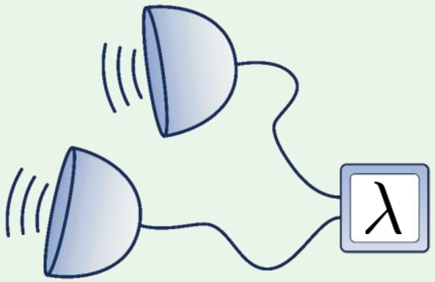
Fixed Measurements

Parametrized state



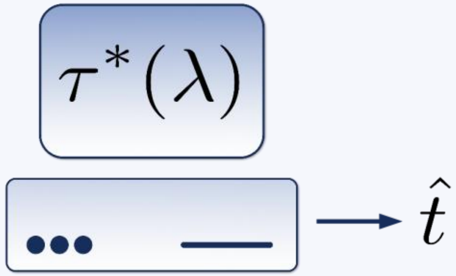
$\rho(t)$

Fixed measurement



$M(\lambda)$

Post-processing



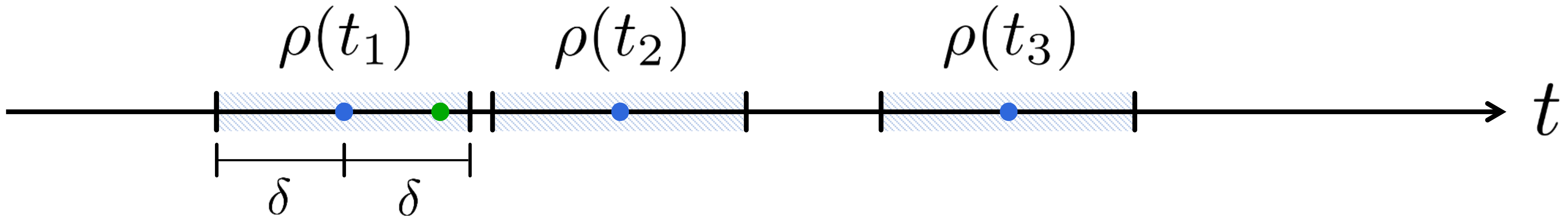
$\hat{t} = \tau^*(\lambda)$

Optimal post-processing is given by the
smoothed maximum a-posteriori estimator

$$\tau^*_{\mathrm{SMAP}}(\lambda) = \underset{t}{\operatorname{argmax}} \int_{t-\delta}^{t+\delta} \mathrm{d}\mu(\tau)\ \mathrm{Tr}[\rho(\tau) M(\lambda)]$$

Connection to Hypothesis Testing

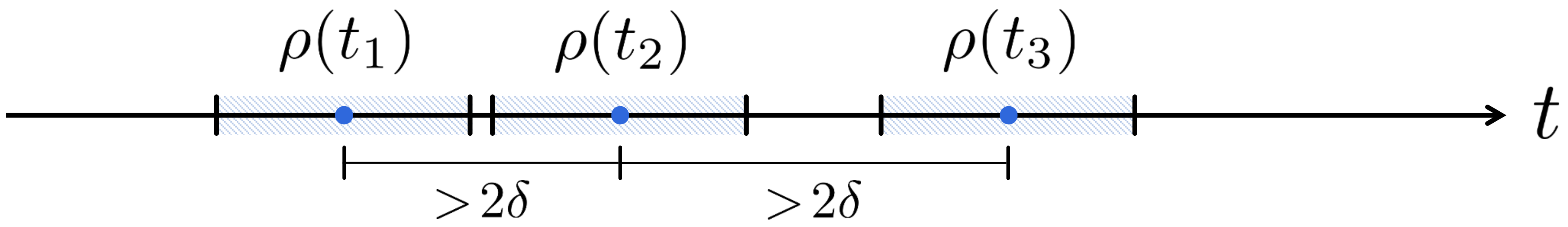
Metrology problem



$Q(\hat{t}) \to \hat{t}$ $\qquad$ $\mathbb{P}[|t_i - \hat{t}| \leq \delta] \geq \overline{\eta}$

Connection to Hypothesis Testing

Multi-hypothesis testing problem



$Q(\hat{t}) \to \hat{t} \to \text{closest } t_i$ $\quad$ $\mathbb{P}[\hat{i} = i] \geq \overline{\eta}$

We conclude that

$$\overline{\eta} \leq \overline{P}_s(\{\rho(t_i)\}) \text{ as long as } |t_i - t_j| > 2\delta$$

The PAC Metrology Framework

$\overline{\eta}$

SUCCESS PROBABILITY

What is the probability of obtaining an estimate within a fixed tolerance?

$\overline{\delta}$

ESTIMATION TOLERANCE

What is the smallest tolerance that still guarantees a fixed success probability?

$\overline{n}$

SAMPLE COMPLEXITY

How many copies of a state do I need to guarantee a fixed success probability and tolerance?

Estimation Tolerance

So far, we analyzed the success probability at fixed tolerance. But in applications, we often care about the achievable precision at fixed success probability.

Minimax estimation tolerance

$$\bar{\delta}(\bar{\eta}) = \inf \left\{ \delta' \geq 0 \;\middle|\; \bar{\eta} \leq \min_t \int_{-\delta'}^{\delta'} \mathrm{d}\tau \; \mathrm{Tr}[\rho(t) Q(t+\tau)] \right\}$$

Finite-sample Cramér-Rao bound

Cramér-Rao bound

$$\sigma(\hat{t}) \geq \frac{1}{\sqrt{\mathcal{F}(t)}}$$

Our bound

$$\overline{\delta} \geq \frac{O\left(\sqrt{\log \frac{1}{1-\overline{\eta}}} - \boxed{q} \log \frac{1}{1-\overline{\eta}}\right)}{\sqrt{\min_t \mathcal{F}(t)}}$$

In the i.i.d. case

$$q = O\left(\frac{1}{\sqrt{n}}\right)$$

Sample Complexity

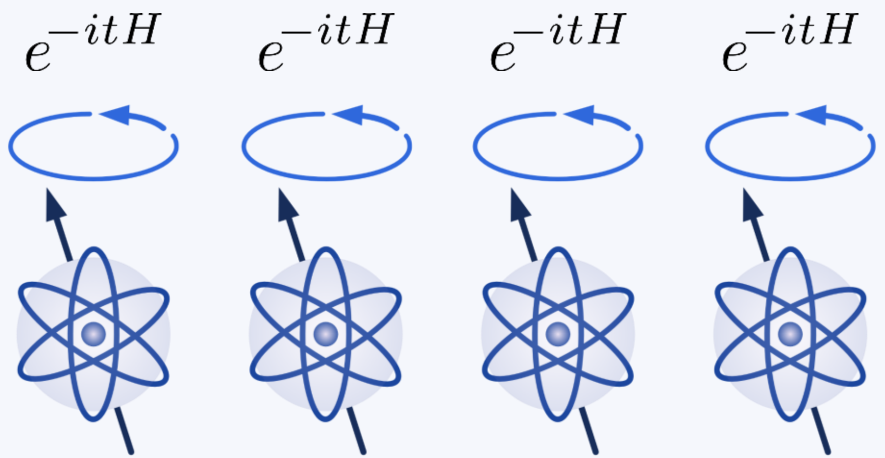
What if we care about both the achievable precision and the success probability? Then we have to ask how many copies of a state we need to achieve it.

Minimax sample complexity

$$\overline{n}(\overline{\eta}, \overline{\delta}) = \min \left\{ n' \in \mathbb{N} \,\middle|\, \overline{\eta} \leq \min_{t} \int_{-\overline{\delta}}^{\overline{\delta}} \mathrm{d}\tau \; \mathrm{Tr}[\rho^{\otimes n'}(t) Q_{n'}(t+\tau)] \right\}$$

Phase estimation

Local evolution of
an ensemble of spins under
the same phase Hamiltonian



For the regular phase Hamiltonian and $t \in [0, 2\pi)$
this yields a **covariant** set of states

Optimal Measurement

We show that the **pretty good measurement** is optimal for covariant state sets

We use this result to obtain a closed-form solution for the minimax success probability

$$\overline{\eta}^*(\delta, \psi) = \sum_{\lambda,\lambda'} |\psi_\lambda||\psi_{\lambda'}| \frac{\sin(\delta(\lambda - \lambda'))}{\pi(\lambda - \lambda')}$$

$$H = \sum_\lambda \lambda \Pi_\lambda$$

$$|\psi\rangle = \sum_\lambda \Pi_\lambda |\psi\rangle = \sum_\lambda \psi_\lambda |\psi_\lambda\rangle$$

Comparison of Probe States

The closed-form solution factilitates a numerical comparison of different probe states

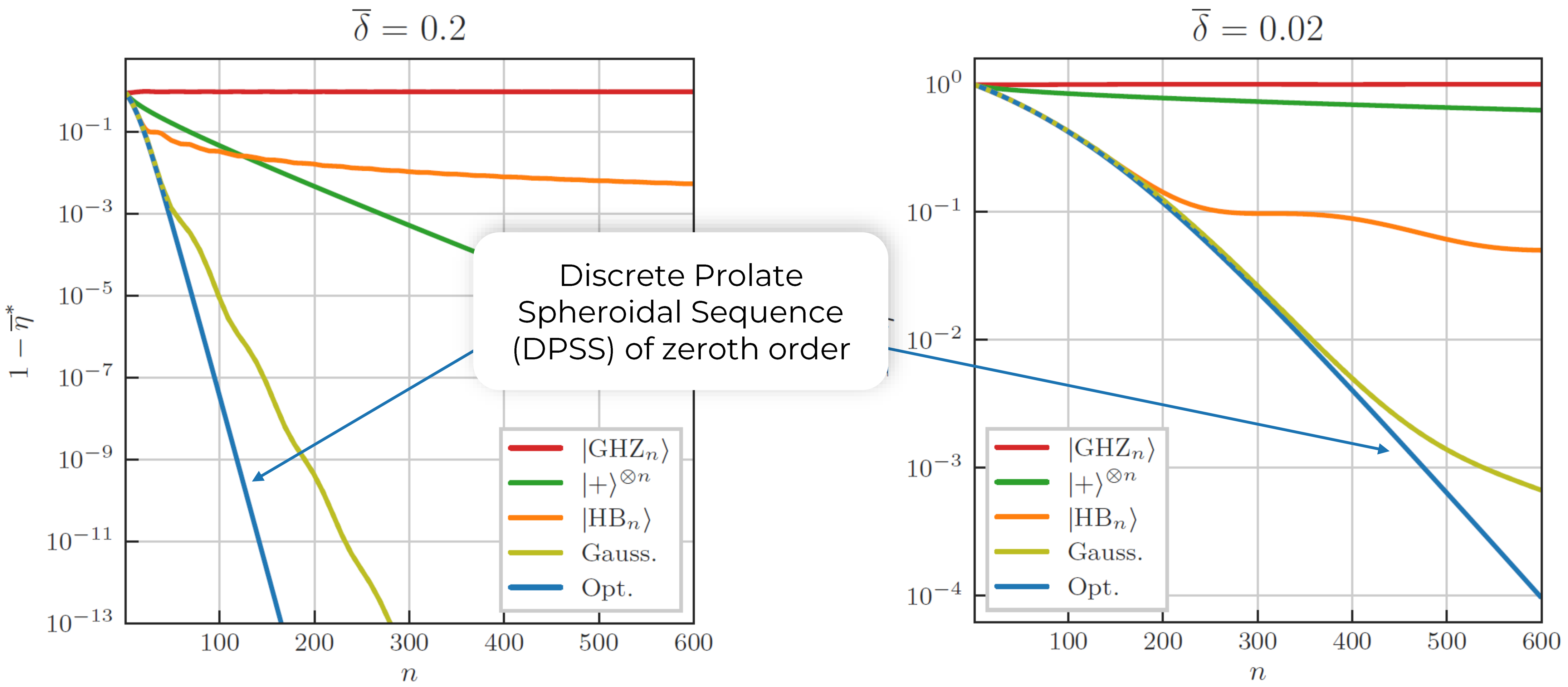
GHZ

$$|\mathrm{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |n\rangle)$$

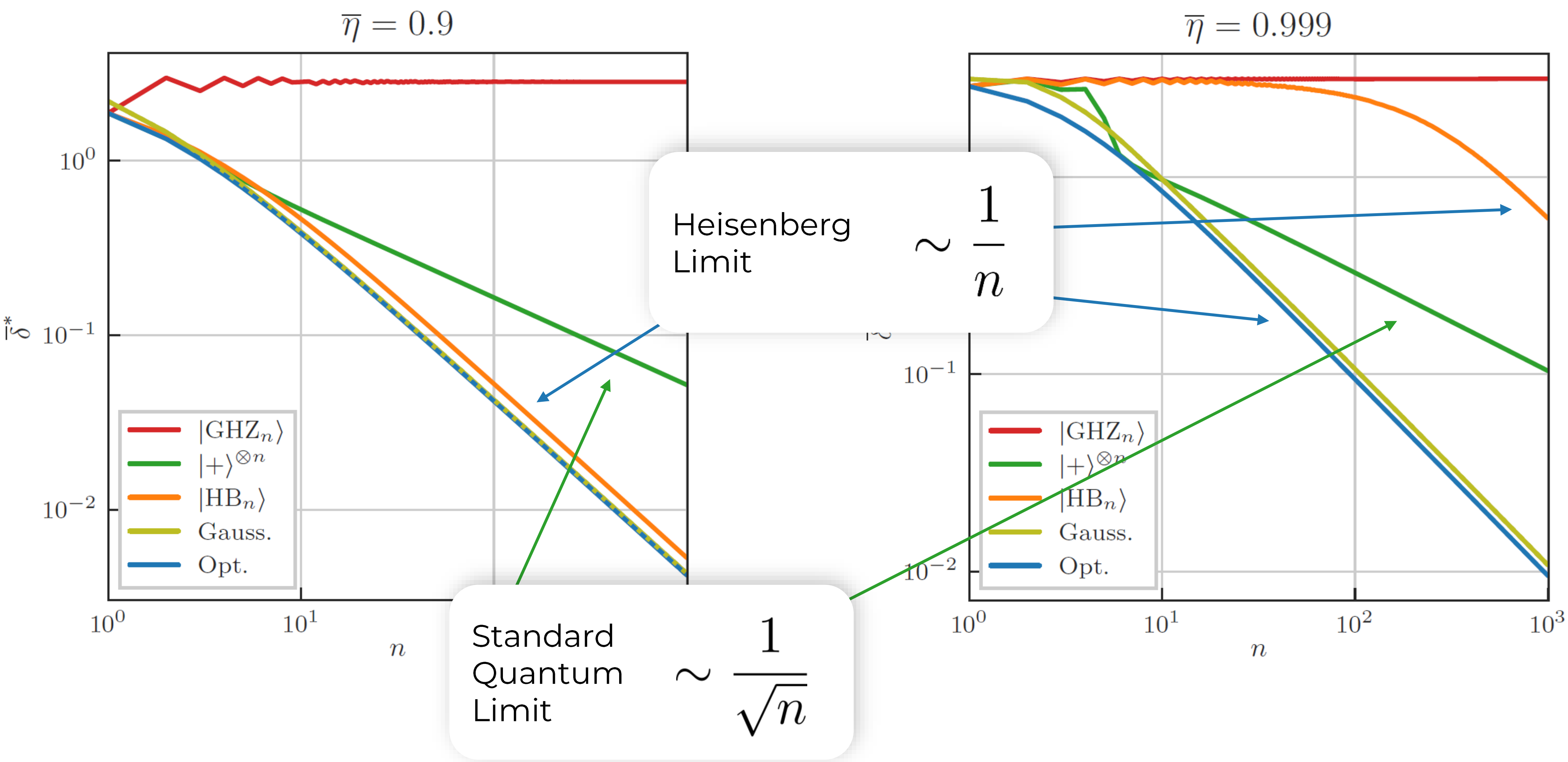
Holland-Burnett

$$|\mathrm{HB}_n\rangle = \frac{1}{\sqrt{n+1}}(|0\rangle + |1\rangle + |2\rangle + \cdots + |n\rangle)$$

Success Probability

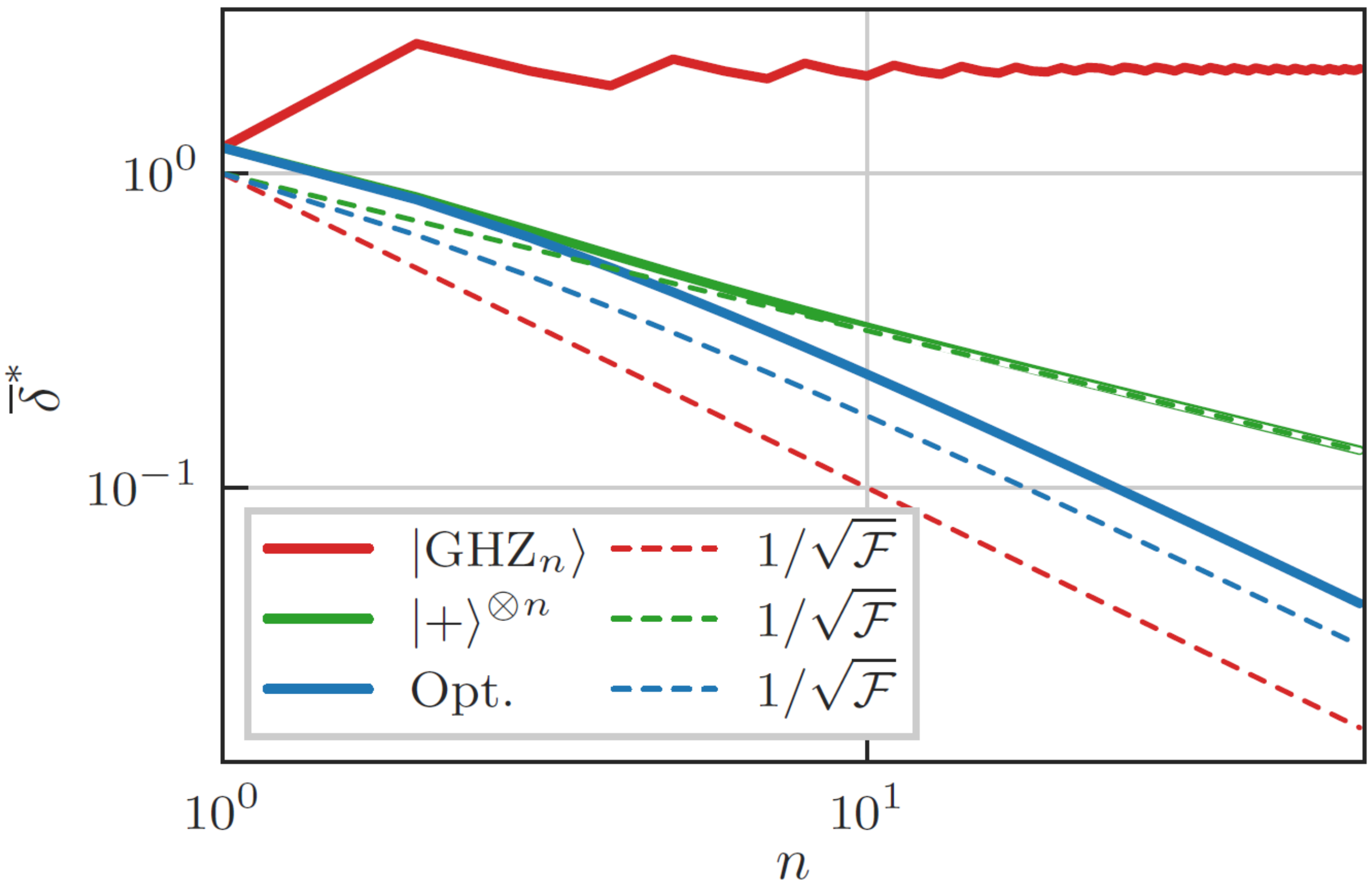


Estimation Tolerance



Comparison with QCRB

$$\overline{\eta} = \mathrm{erf}(1/\sqrt{2})$$



Further Results in the Paper

- We connect our quantities to single-shot entropy measures
- We lift the hypothesis testing connection to quantum channels with different access models
- We discuss many possible extensions of our results and definitions, e.g. the multi-parameter case
- We give an overview of open questions

Open Questions

- What measurements (i.e. POVMs) give good out-of-the-box performance guarantees? Pretty good measurement?
- Improved finite-sample analogues of the Cramér-Rao bound
- Understanding the advantages of adaptive processing and entanglement
- What are the admissible scalings with mixed asymptotics?
- How do noise and error correction fit into this picture?

Summary

- We give new tools to understand quantum metrology in the single-shot regime
- Our framework is very close to quantum information theory both in tools as in results
- A plethora of open questions ranging from practically oriented to completely information-theoretic
- An exciting opportunity to explore new directions in quantum metrology!

Thank you for your attention!

Slides

arXiv:2307.06370

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