Quantum metrology in the finite-sample regime

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Quantum metrology in the finite-sample regime

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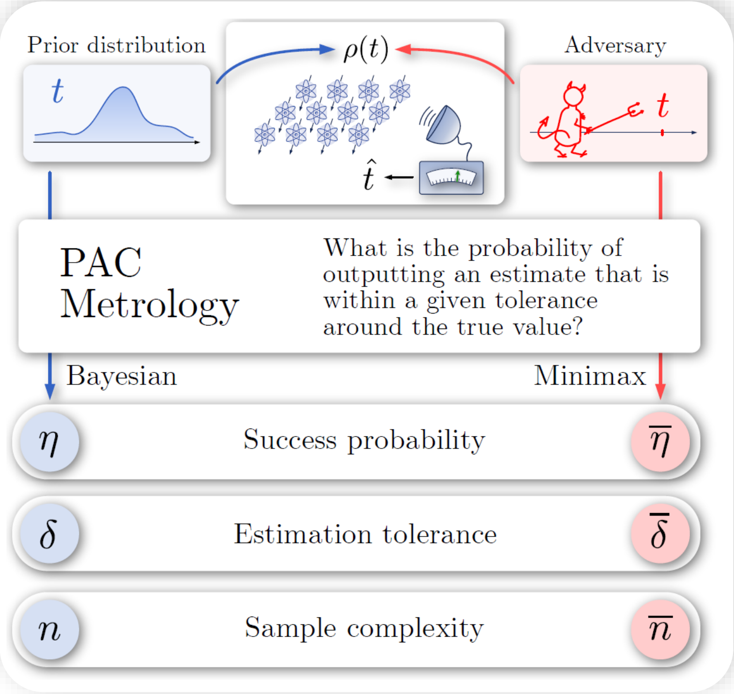
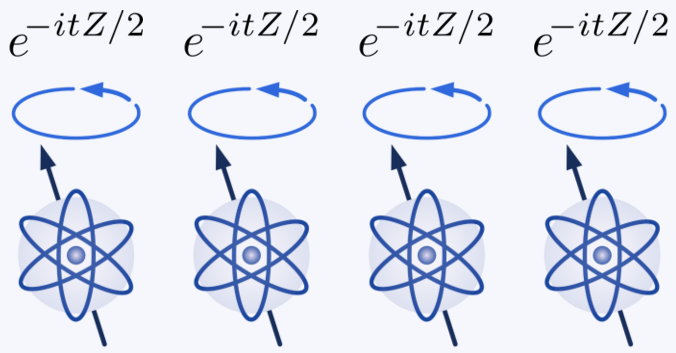


Quantum Metrology

GOAL
Devise a protocol that estimates the phase as well as possible

PHASE ESTIMATION
Local evolution of an ensemble of spins under the phase Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} Z_i$$



LIMITATIONS OF PRACTICAL METROLOGY
- Low numbers of quantum systems
- Slow operation speed of certain platforms
- Drift of the system parameters

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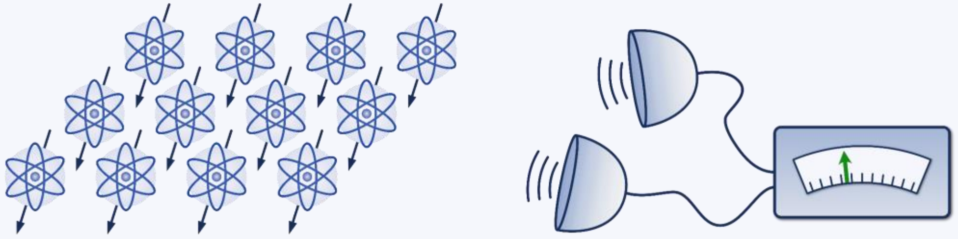
Practical metrology happens in the **finite-sample regime!**

Traditional Quantum Metrology



$\rho(t)$ $Q(\hat{t})$

FIGURE OF MERIT

- **Mean squared error** of the estimate
- Equal to the **variance** if the estimate is exact in expectation

QUANTUM FISHER INFORMATION

$$F(\rho(t), \rho(t+\tau)) =: 1 - \frac{1}{4}\mathcal{F}(t)\tau^2 + O(\tau^3)$$

CRAMÉR-RAO BOUND[1]

$$\mathrm{Var}(\hat{t}) \geq \frac{1}{\boxed{\mathcal{F}(t)}}$$

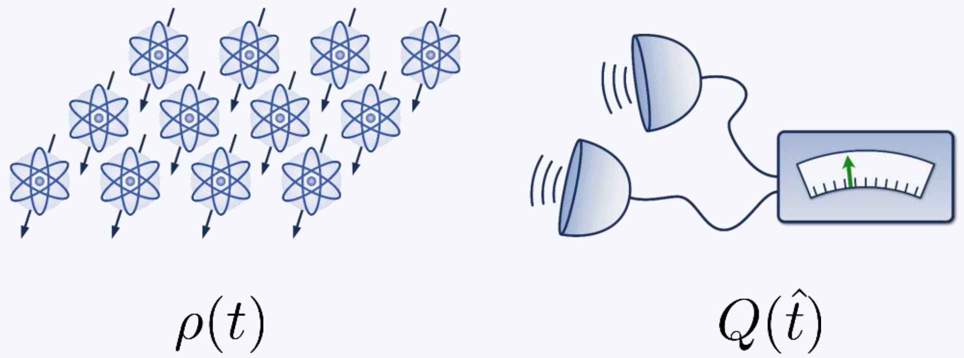
WHAT ABOUT THE FINITE-SAMPLE REGIME?

- Cramér-Rao bound still constrains precision
- But it can be **overly optimistic**

Optimizing for large Fisher information can lead to poor finite-sample performance

[1]Helstrom, *Phys. Lett. A* (1967)

Single-shot Quantum Metrology



SINGLE-SHOT FIGURE OF MERIT

- Define an **estimation tolerance** δ that should be achieved
- Determine the probability of estimating the parameter within that tolerance

SUCCESS PROBABILITY[1]

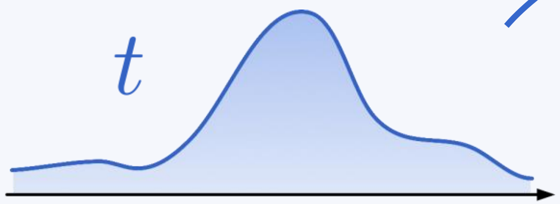
$$\boxed{\mathbb{P}}[|t - \hat{t}| \leq \delta]$$

We need a way to assign probabilities!

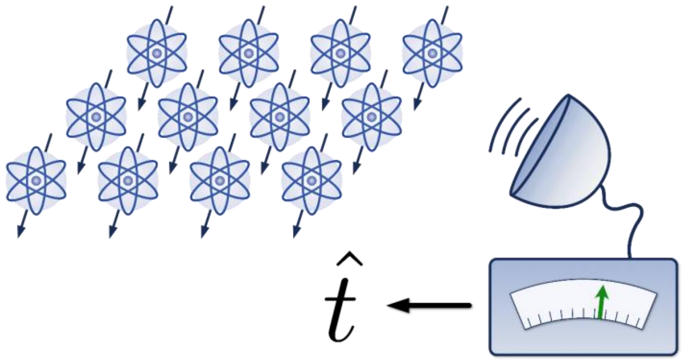
[1]See also Hayashi, *J. Phys. A* (2002), Walter and Renes, *IEEE Trans. Inform. Theory* (2014)

Single-shot Quantum Metrology

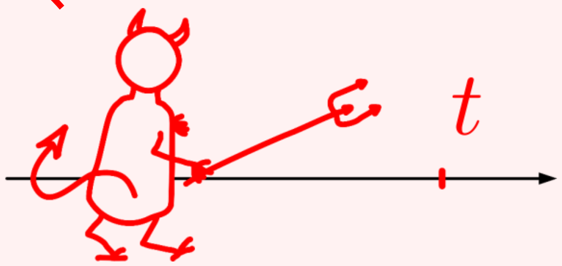
Prior distribution



$\rho(t)$



Adversary



Bayesian success probability

$$\eta := \int \mathrm{d}\mu(t) \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)]$$

Minimax success probability

$$\overline{\eta} := \min_{t} \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)]$$

Optimal Success Probability

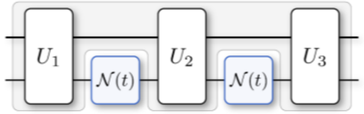
OPTIMAL SUCCESS PROBABILITY

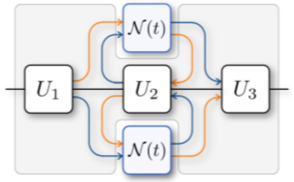
Find the optimal way of estimating a parameter

$$\overline{\eta}^* := \boxed{\max_{Q(\hat{t})}} \left\{ \min_{t} \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)] \right\}$$

Constitutes a **semi-infinite program**[1] (think: a continuous semi-definite program)

Similarly for **general protocols** involving quantum channels



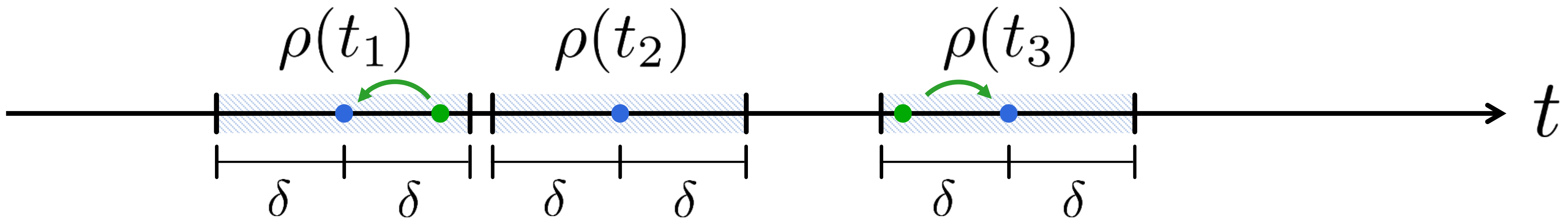


[1]Hettich and Kortanek, *SIAM Review* (1993)

Hypothesis Testing Bound

MOTIVATION
If states that are $O(\delta)$ apart are hard to distinguish, estimating the parameter to precision δ should also be hard

Perform a reduction from quantum metrology to multi-hypothesis testing



We can use the metrology protocol to solve the multi-hypothesis testing task

THEOREM
The success probability of quantum metrology cannot exceed the success probability of distinguishing states at times that are at least 2δ apart

The Single-Shot Metrology Framework

$\overline{\eta}$

SUCCESS PROBABILITY[1]

What is the probability of obtaining an estimate within a fixed tolerance?

$\overline{\delta}$

ESTIMATION TOLERANCE[2]

What is the smallest tolerance that still guarantees a fixed success probability?

$\overline{n}$

SAMPLE COMPLEXITY

How many copies of a state do I need to guarantee a fixed success probability and tolerance?

[1]See also Hayashi, *J. Phys. A* (2002), Walter and Renes, *IEEE Trans. Inform. Theory* (2014) [2]See also Yang et al., Proc. R. Soc. A (2018)

Single-shot Cramér-Rao Bound

MOTIVATION
The Cramér-Rao bound also constrains precision in the finite-sample case. Can we find a comparable bound in our framework?

CRAMÉR-RAO BOUND

$$\mathrm{Std}(\hat{t}) = \sqrt{\mathrm{Var}(\hat{t})} \geq \frac{1}{\sqrt{\mathcal{F}(t)}}$$

Quantifies how far we are from the asymptotic limit

In the i.i.d. case: $q = O\left(\frac{1}{\sqrt{n}}\right)$

SINGLE-SHOT CRAMÉR-RAO BOUND

Due to worst-case nature of the minimax estimation tolerance

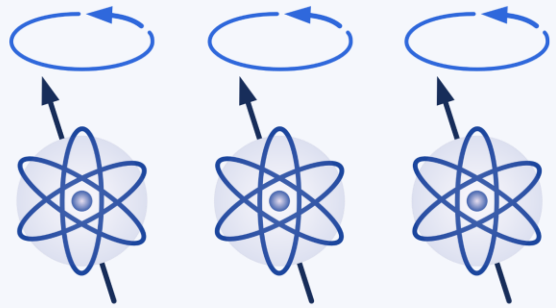
$$\bar{\delta} \geq \frac{O\left(\sqrt{\log\frac{1}{1-\bar{\eta}}} - \boxed{q}\log\frac{1}{1-\bar{\eta}}\right)}{\sqrt{\boxed{\min_t}\mathcal{F}(t)}}$$

Optimal Phase Estimation

PHASE ESTIMATION

Local evolution of an ensemble of spins under the phase Hamiltonian

$$e^{-itZ/2} \quad e^{-itZ/2} \quad e^{-itZ/2}$$



$$H = \frac{1}{2}\sum_{i=1}^{n} Z_i$$

We show that the **pretty good measurement**[1] is optimal for covariant state sets

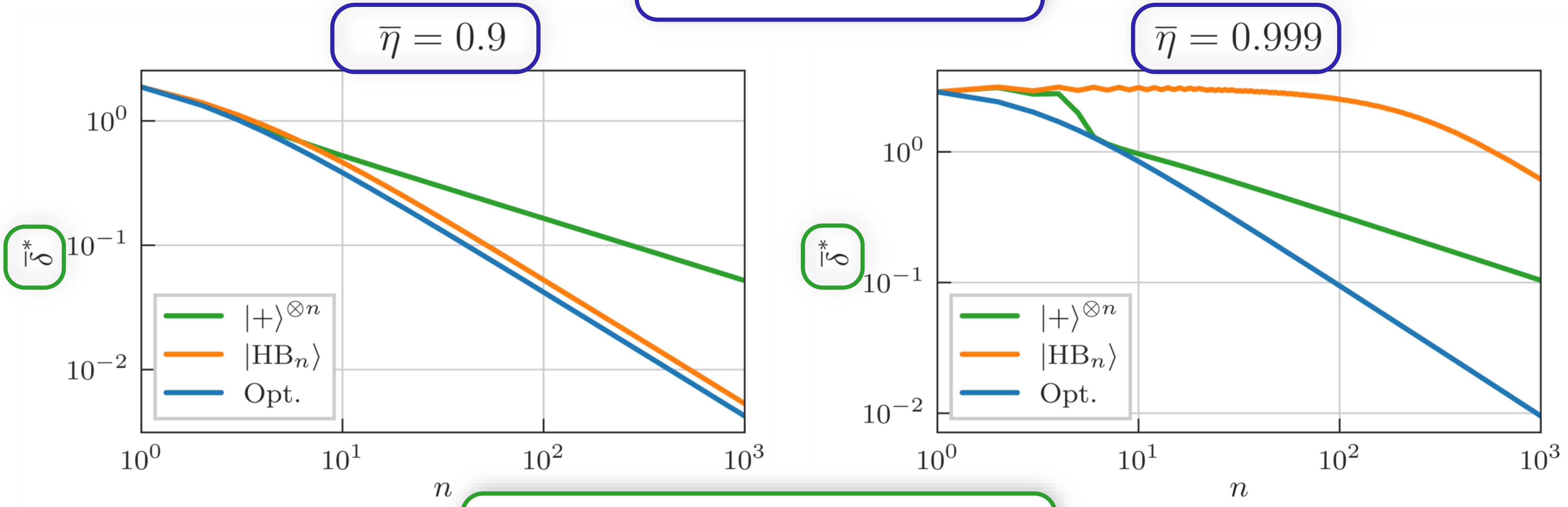
We use this to obtain a **closed-form solution** for the minimax success probability

The closed-form solution factilitates a **numerical comparison of** different **probe states**

[1]Holevo, *Rep. Math. Phys.* (1997)

Minimax Estimation Tolerance

Desired probability of success



Achievable optimal tolerance (Loglog plot)

Minimax Esti

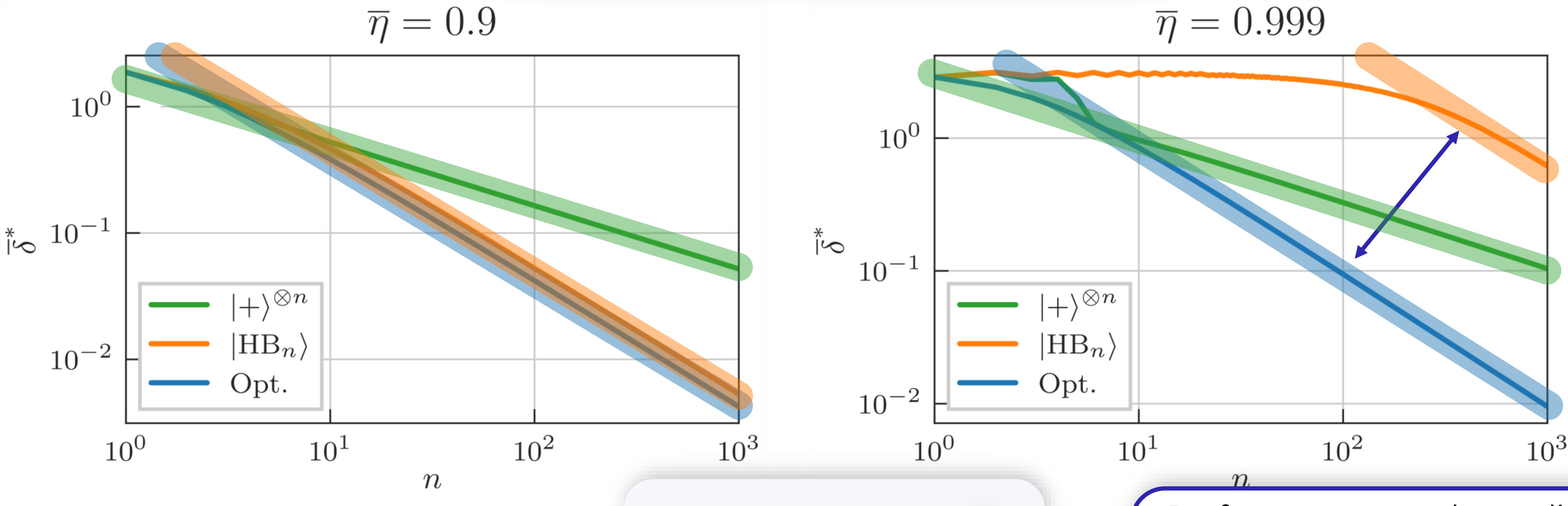
Standard Quantum Limit $\sim \frac{1}{\sqrt{n}}$

Holland-Burnett State[1]

$$|\mathrm{HB}_n\rangle = \frac{1}{\sqrt{n+1}}(|0\rangle + |1\rangle + |2\rangle + \cdots + |n\rangle)$$

Has similar Fisher information as optimal state

Heisenberg Limit $\sim \frac{1}{n}$



Heisenberg Limit $\sim \frac{1}{n}$

Performance gap depending on the success probability

Cannot be predicted by Fisher information!

[1]Holland and Burnett, *Phys. Rev. Lett.* (1993)

Summary

Understanding quantum metrology in the single-shot regime requires tools beyond the Cramér-Rao bound

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We change our perspective from quantifying estimation variances to success probabilities

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Allows to rigorously study the single-shot regime and gives a strong connection to quantum information theory

Opens up many **exciting directions** in a field that many considered “solved“!

What protocols give good out-of-the-box guarantees?

How do noise and error correction affect the single-shot performance?

How much can we gain with adaptive processing and entanglement?

Let us explore new directions in quantum metrology!

Thank you for your attention!

Slides

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