Tomography of parametrized quantum states

Quantinuum Research Seminar

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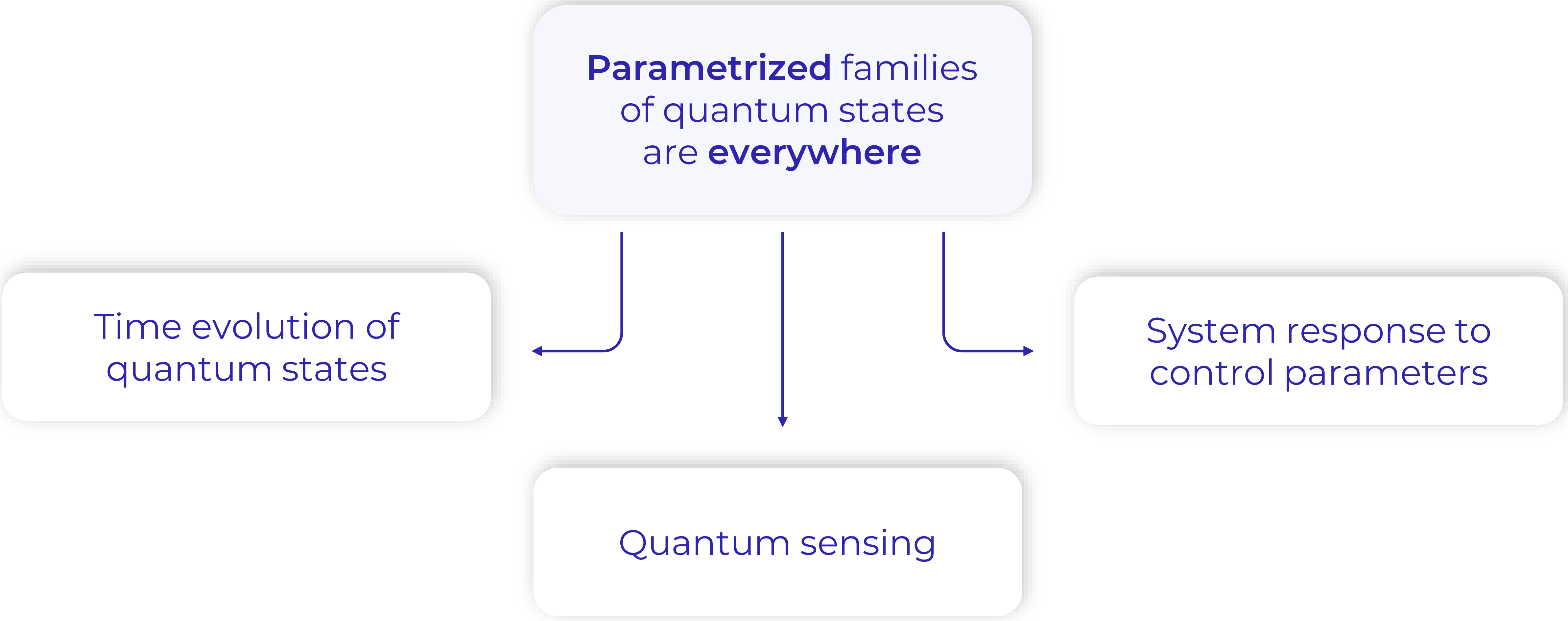
Tomography

Is my quantum computer doing what I think it is doing?

Can I convince others that my quantum computer is doing what I claim it is doing?

Parametrized Quantum States



Parametrized families of quantum states are **everywhere**

Time evolution of quantum states

Quantum sensing

System response to control parameters

How can we **formalize** and **perform** tomography of parametrized states?

What is a parametrized quantum state?

Parametrized Quantum States

FUNCTIONS AND NORMS

Inner Product

$$\langle f, g\rangle := \int \mathrm{d}\mu(\boldsymbol{x})\, f^*(\boldsymbol{x}) g(\boldsymbol{x})$$

Norm

$$\|f\|_p^p := \int \mathrm{d}\mu(\boldsymbol{x})\, |f(\boldsymbol{x})|^p$$

Orthonormal Basis

$$\{\varphi_k\}_k : \langle \varphi_k, \varphi_{k'}\rangle = \delta_{kk'}$$

PARAMETRIZED STATES

$$\rho(\boldsymbol{x}) = \sum_k \alpha_k \varphi_k(\boldsymbol{x})$$

$$\alpha_k = \langle \varphi_k, \rho(\cdot)\rangle$$

Parametrized Quantum States

FOURIER BASIS

$$\varphi_k(t) = e^{-ikt}, k \in \mathbb{Z}$$

$$t \in [0, 2\pi), \mu(t) = \frac{1}{2\pi}$$

HAMILTONIAN EVOLUTION

$$\rho(t) = e^{-iHt}\rho_0 e^{iHt} \qquad H = \sum_\lambda \lambda \Pi_\lambda$$

$$\rho(t) = \sum_{k\in\mathbb{Z}} \alpha_k e^{-ikt}$$

$$\Rightarrow \; \alpha_k = \sum_\lambda \Pi_\lambda \rho_0 \Pi_{\lambda+k}$$

What is Tomography?

Tomography

QUANTUM STATE TOMOGRAPHY

Given access to multiple copies of a quantum state, produce an estimate with high probability that is close in trace distance

$$\mathbb{P}\left[\|\rho - \hat{\rho}\|_1 \leq \epsilon\right] \geq 1 - \delta$$

Exponential sample complexity

SHADOW TOMOGRAPHY

Given access to multiple copies of a quantum state, produce an estimate with high probability that approximately reproduces a set of expectation values

$$\mathbb{P}\left[\max_i |\mathrm{Tr}[(\rho - \hat{\rho})O_i]| \leq \epsilon\right] \geq 1 - \delta$$

Polynomial sample complexity*

*For interesting sets of observables

General Tomographic Procedures

INDUCED SEMINORM

For a given set of observables $\mathcal{O}$, we define the *induced seminorm*

$$\|X\|_{\mathcal{O}} := \sup_{O\in\mathcal{O}} |\operatorname{Tr}[XO]|$$

EXAMPLE

Local Clifford Shadow Tomography is a tomographic procedure relative to local observables

$$\mathcal{O}_\ell := \{O : \|O\|_\infty \leq 1, O \text{ is } \ell\text{-local}\}$$

with sample complexity

$$T(\epsilon,\delta,n) = O\left(\frac{\ell 12^\ell}{\epsilon^2}\log\frac{n}{\delta}\right)$$

TOMOGRAPHIC PROCEDURE

An experimental procedure is an (ϵ,δ,n)-*tomographic procedure* relative to a set of observables $\mathcal{O}$ if it guarantees

$$\mathbb{P}\left[\|\rho-\hat{\rho}\|_{\mathcal{O}} \leq \epsilon\right] \geq 1-\delta$$

The necessary number of copies $T(\epsilon,\delta,n)$ is the *sample complexity* of the tomographic procedure

Figure of Merit for Parametrized States

INDUCED SEMINORM

For a given set of observables $\mathcal{O}$, we define the *induced seminorm*

$$\|X\|_{\mathcal{O}} := \sup_{O \in \mathcal{O}} |\operatorname{Tr}[XO]|$$

FUNCTION p-NORM

For a given measure $\mu(\boldsymbol{x})$, we define the p-norm of a function

$$\|f\|_p^p = \int \mathrm{d}\mu(\boldsymbol{x}) \, |f(\boldsymbol{x})|^p$$

Largest norm of the parametrized expectation value

INDUCED L^p SEMINORM

The induced L^p seminorm of a parametrized operator

$$\boldsymbol{x} \mapsto X(\boldsymbol{x})$$

is given by

$$\|\!|X(\cdot)|\!\|_{\mathcal{O},p}^p = \sup_{O \in \mathcal{O}} \int \mathrm{d}\mu(\boldsymbol{x}) \, |\operatorname{Tr}[O X(\boldsymbol{x})]|^p$$

Figure of Merit for Parametrized States

INDUCED SEMINORM

For a given set of observables $\mathcal{O}$, we define the *induced seminorm*

$$\|X\|_{\mathcal{O}} := \sup_{O \in \mathcal{O}} |\operatorname{Tr}[XO]|$$

VECTOR p-NORM

We define the p-norm of a d-dimensional vector

$$\|\boldsymbol{x}\|_p^p = \sum_{i=1}^{d} |x_i|^p$$

Largest norm of the vector of expectation values

INDUCED lᵖ SEMINORM

The induced lᵖ seminorm of a vector of operators

$$\boldsymbol{X} = (X_1, X_2, \ldots, X_d)$$

is given by

$$\|\!|\boldsymbol{X}|\!\|_{\mathcal{O},p}^p := \sup_{O \in \mathcal{O}} \sum_{i=1}^{d} |\operatorname{Tr}[OX_i]|^p$$

Useful Properties

SUBMULTIPLICATIVITY
For a vector of operators, we define matrix-vector multiplication in the usual sense. Then,

$$\|\!|A\boldsymbol{X}|\!\|_{\mathcal{O},p} \leq \|A\|_{p\to p} \|\!|\boldsymbol{X}|\!\|_{\mathcal{O},p}$$

PARSEVAL'S THEOREM
For a parametrized operator given by

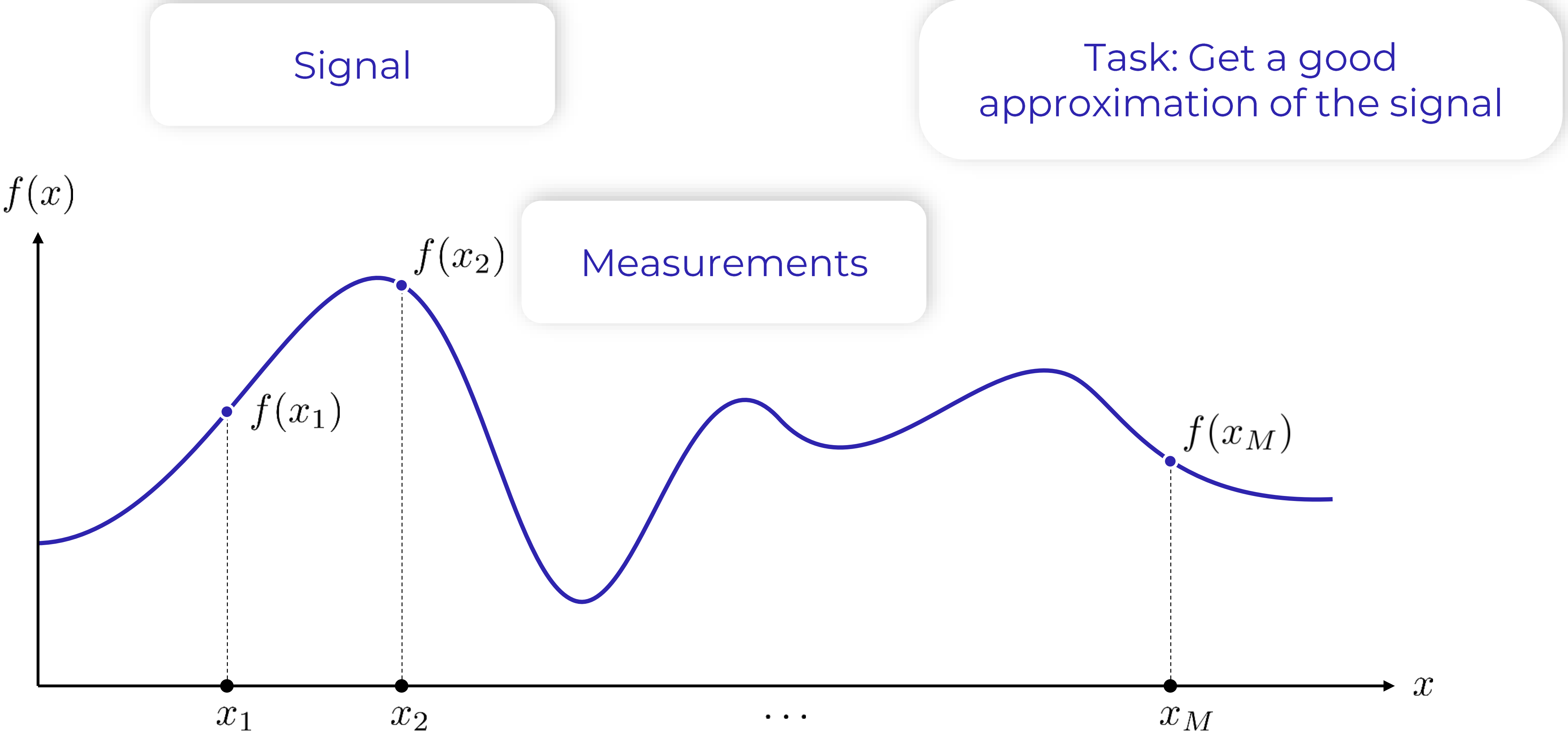
$$X(\boldsymbol{x}) = \sum_k \alpha_k \varphi_k(\boldsymbol{x})$$

we have the identity

$$\|\!|X(\cdot)|\!\|_{\mathcal{O},2} = \|\!|\boldsymbol{\alpha}|\!\|_{\mathcal{O},2}$$

How do we attack this algorithmically?

Signal Processing



Signal

Task: Get a good approximation of the signal

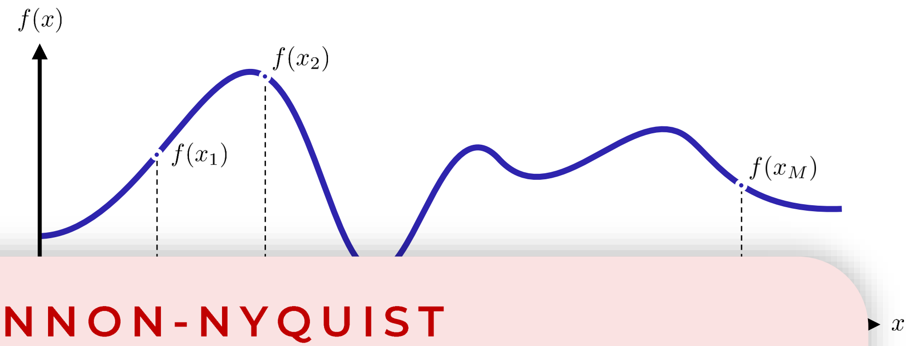
Measurements

Signal Processing

STANDARD APPROACH

Assume finite bandwidth (i.e. maximum frequency)

$$f(x) = \sum_{k=-k_0}^{k_0} c_k e^{-ikx} = \sum_{k\in\Lambda} c_k \varphi_k(x)$$



SHANNON-NYQUIST

You need as many measurements as basis functions

$$M \geq O(D)$$

LINEAR SYSTEM

Measurements form a linear system of equations

$$\begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_M) \end{pmatrix} = \begin{pmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \dots & \varphi_D(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \dots & \varphi_D(x_2) \\ \vdots & \vdots & & \vdots \\ \varphi_1(x_M) & \varphi_2(x_M) & \dots & \varphi_D(x_M) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_D \end{pmatrix}$$

$$\boldsymbol{f} = A\boldsymbol{c} \Rightarrow \boldsymbol{c} = A^{+}\boldsymbol{f}$$

Compressed Sensing

SPARSITY

A signal is sparse if there exists a set $S \subset \Lambda$ such that

$$f(x) = \sum_{k \in S} c_k e^{-ikx}$$

COMPRESSED SENSING

For sparse signals with $s = |S|$, we only need

$$M \geq \tilde{O}(s \log D)$$

HOW?

We find the solution of the linear system with the smallest 1 norm

$$\min \|\boldsymbol{c}\|_1 \text{ s.t. } \boldsymbol{f} = A\boldsymbol{c}$$

This works, essentially, if there are no sparse vectors in the kernel of A, as any vector $\boldsymbol{v} \in \ker A$ can be added to a given solution.

WHERE DO WE MEASURE?

Randomly choosing points according to the measure of orthgonality, $\mu(x)$, guarantees that A has this property with high probability

Algorithm for full recovery

Full Recovery of Parametrized Quantum States

Input ϵ, δ

Set $M = O\left(D \log D/\delta\right), \ \epsilon' = O(\epsilon), \ \delta' = O(\delta/M)$

For $1 \leq i \leq M$

Sample $\boldsymbol{x}_i \sim \mu$

Perform Tomography $\hat{\rho}(\boldsymbol{x}_i) = \textbf{tomographic procedure}(\rho(\boldsymbol{x}_i), \epsilon', \delta')$

Construct Matrix $A_{i,k} = \varphi_k(\boldsymbol{x}_i)$

Full Recovery of Parametrized Quantum States

Input ϵ, δ

Have $A, \hat{\boldsymbol{\rho}} = (\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_M)$

Compute Pseudoinverse A^+

Compute $\hat{\boldsymbol{\alpha}} = A^+ \hat{\boldsymbol{\rho}}$

Output $\hat{\rho}(\cdot) = \sum_{k=1}^{D} \hat{\alpha}_k \varphi_k(\cdot)$

GUARANTEE

The output of the algorithm satisfies

$$\mathbb{P}\left[|||\rho(\cdot) - \hat{\rho}(\cdot)|||_{\mathcal{O},2} \leq \epsilon \right] \geq 1 - \delta$$

and can be combined with any tomographic procedure.

Algorithm for sparse recovery

Sparse Parametrized Quantum States

SPARSE PARAMETRIZED STATES

A parametrized quantum state

$$\rho(\boldsymbol{x}) = \sum_k \alpha_k \varphi_k(\boldsymbol{x})$$

is (s, γ_{ℓ^p})-sparse if there exists a set S of cardinality $|S| = s$ such that

$$|||\boldsymbol{\alpha} - \boldsymbol{\alpha}_S|||_{\mathcal{O},p} \leq \gamma_{\ell^p}$$

Idea: Split the task into two

Support Identification

Sparse Recovery

Sparse Recovery

Essentially the same algorithm, taking the pseudoinverse over the set S and only performing tomography at

$$M = \tilde{O}\left((s/\Delta^2)(\log D + \log 1/\delta)\right)$$

parameter values

GUARANTEE

The output of the algorithm satisfies

$$\mathbb{P}\left[|||\rho(\cdot) - \hat{\rho}(\cdot)|||_{\mathcal{O},2} \leq \epsilon + \gamma_{\ell^2} + \Delta\gamma_{\ell^1}\right] \geq 1 - \delta$$

and can be combined with any tomographic procedure.

Support Identification

Idea: Take a random observable, use classical compressed sensing on the parametrized expectation value to identify support

Trivially works for exact sparsity

General statements are difficult to come by, as the „sparsity pattern“ for a given observable need not match the true pattern

We can give guarantees under additional assumptions!

Shadow tomography of free Fermionic time evolution

Setup

FREE FERMIONIC HAMILTONIAN

We consider time evolution under a Hamiltonian composed of Majorana operators

$$H = i\sum_{i=1}^{n}\sum_{j=1}^{n} F_{ij}\gamma_i\gamma_j$$

with interaction strength

$$J := \max_{ij} |F_{ij}|$$

TIME EVOLUTION

The time evolution of a state expands into

$$\rho(t) = \sum_{i=1}^{2^n}\sum_{j=1}^{2^n} e^{-i(\lambda_i-\lambda_j)t}\Pi_i\rho_0\Pi_j$$

with bounded frequency

$$|\lambda_i - \lambda_j| \le n^2 J$$

Difficult to approximate in the Fourier basis!

Solution

TIME EVOLUTION

The time evolution of a state expands into

$$\rho(t) = \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} e^{-i(\lambda_i - \lambda_j)t} \Pi_i \rho_0 \Pi_j$$

with bounded frequency

$$|\lambda_i - \lambda_j| \leq n^2 J$$

Chebychev polynomials work due to the bounded frequency!

GUARANTEES

We can recover all ℓ-local observables for time $t \in [-T, T]$, i.e.

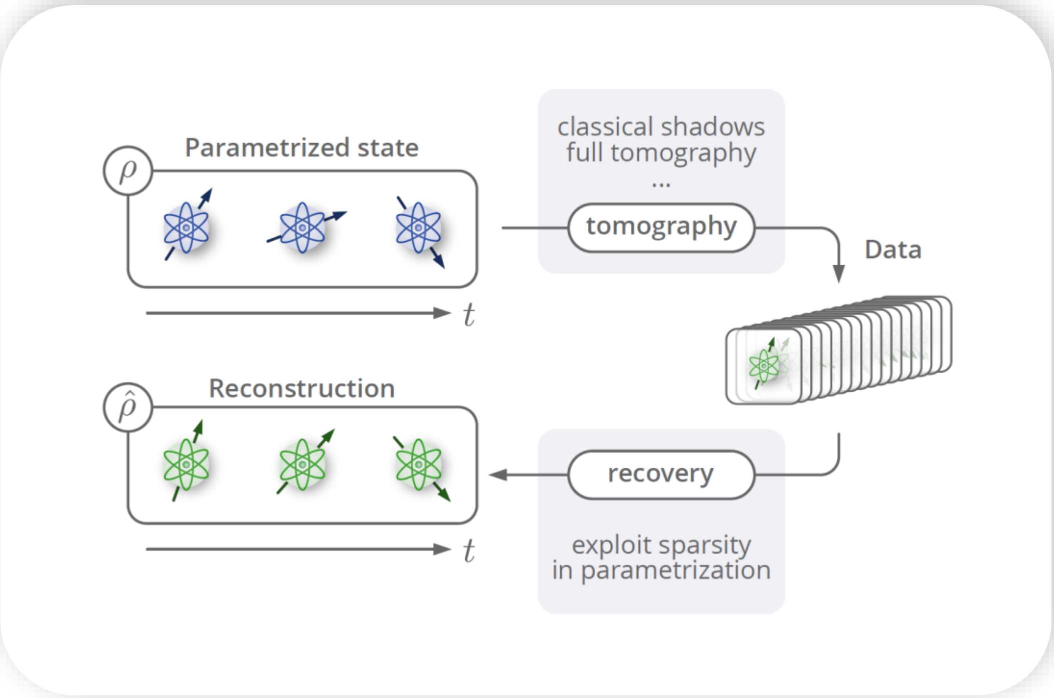
$$\mathbb{P}\left[\| \rho(\cdot) - \hat{\rho}(\cdot) \|_{\mathcal{O}_{\ell},2} \leq \epsilon + 2\gamma \right] \geq 1 - \delta$$

using a total number of copies

$$N = \tilde{O}\left(n^{\ell+2} J T \frac{\ell^{3/2}}{\epsilon^2} \log \frac{D}{\gamma} \log^2 \frac{1}{\delta} \right)$$

Summary

- A formal umbrella for different notions of tomography
- A sensible figure of merit for tomography of parametrized quantum states
- An algorithm that efficiently recovers quantum states with huge gains for sparse parameter dependence
- Algorithm extends to average case tomography and the tomography of parametrized channels



Parametrized quantum states offer many more interesting questions!

Thank you for your attention.

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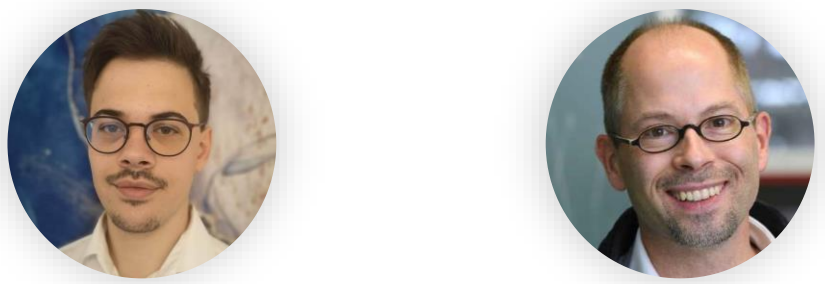

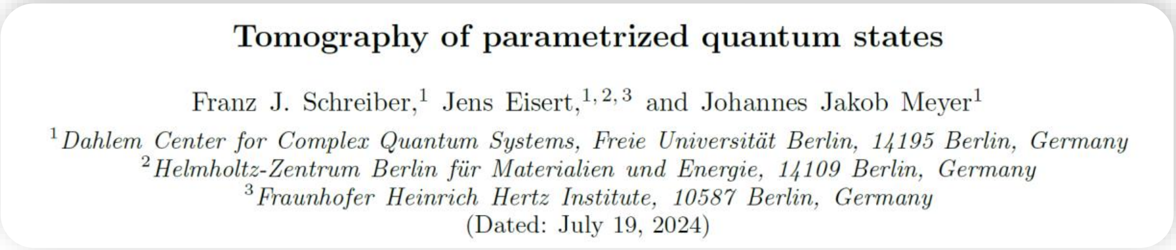

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Article

Slides