Directions in Fundamental Limitations of Quantum Technologies

FLQT Dublin

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Themes of This Talk

Fundamental Limitations
can arise from

Part I	Part II	Part III
Device Noise	Finite Samples	Computational Restrictions

Part I

How bad can **noise** really be?

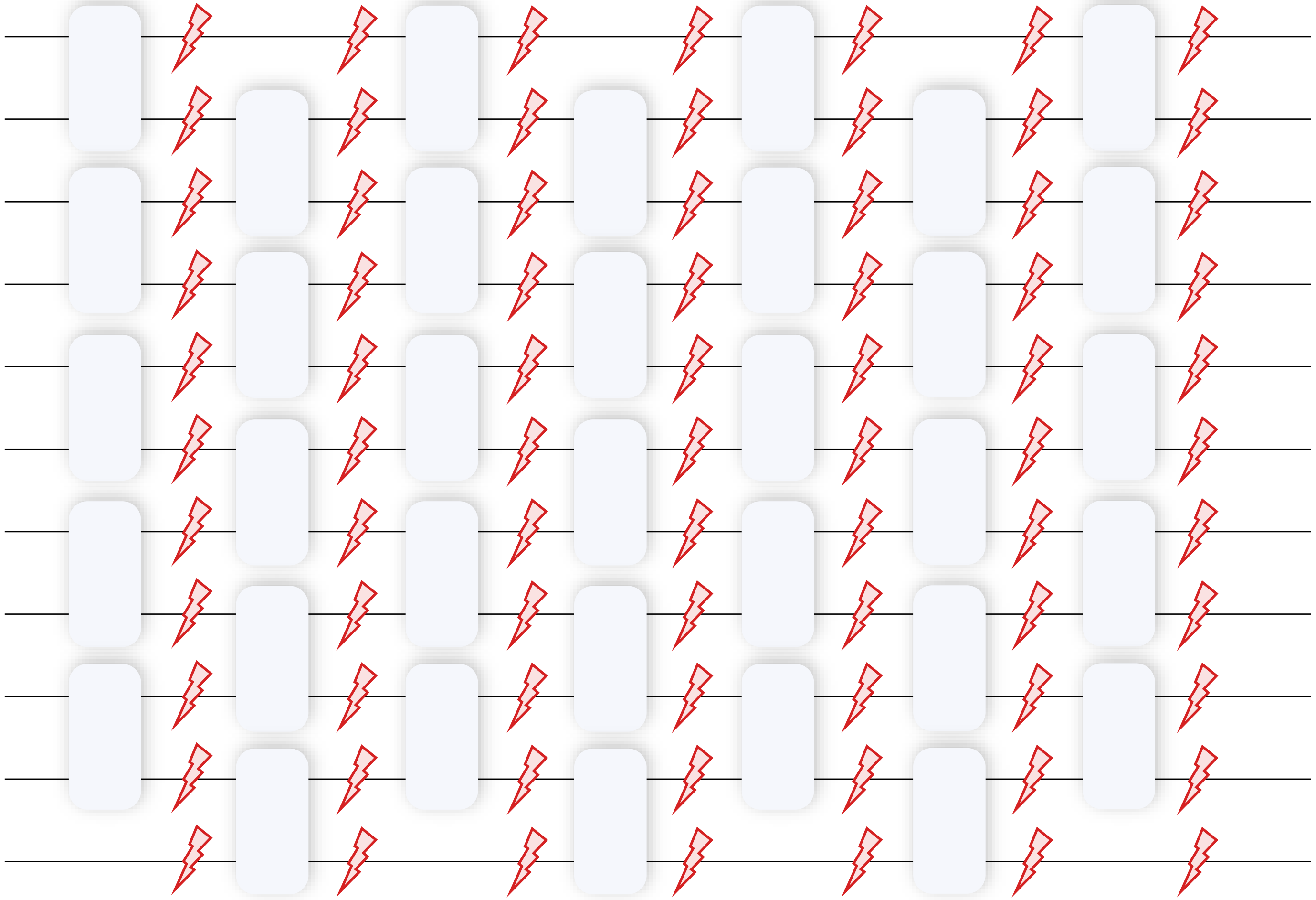
Noisy Quantum Devices

Implementations of quantum operations are imperfect

We can model this using noise channels

Depolarizing Noise

$$\mathcal{D}_\lambda[\rho] = \lambda\rho + (1-\lambda)\omega$$

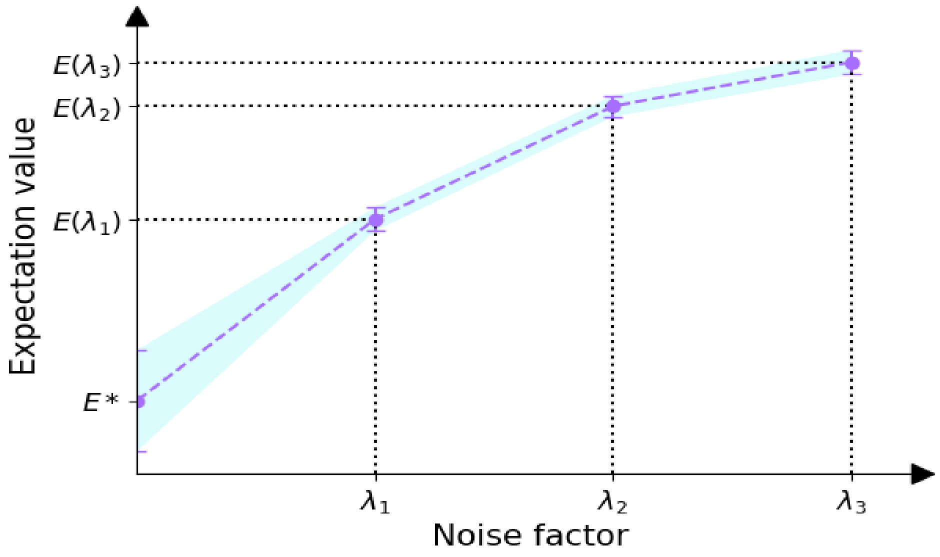


Error Mitigation

KEY IDEA
Can we gather more data and use it to **classically** „undo“ the noise?

ZERO NOISE EXTRAPOLATION[1]

- Increase noise level by adding superfluous operations
- Compute expectation values at different noise levels
- Fit the curve and extrapolate to zero noise



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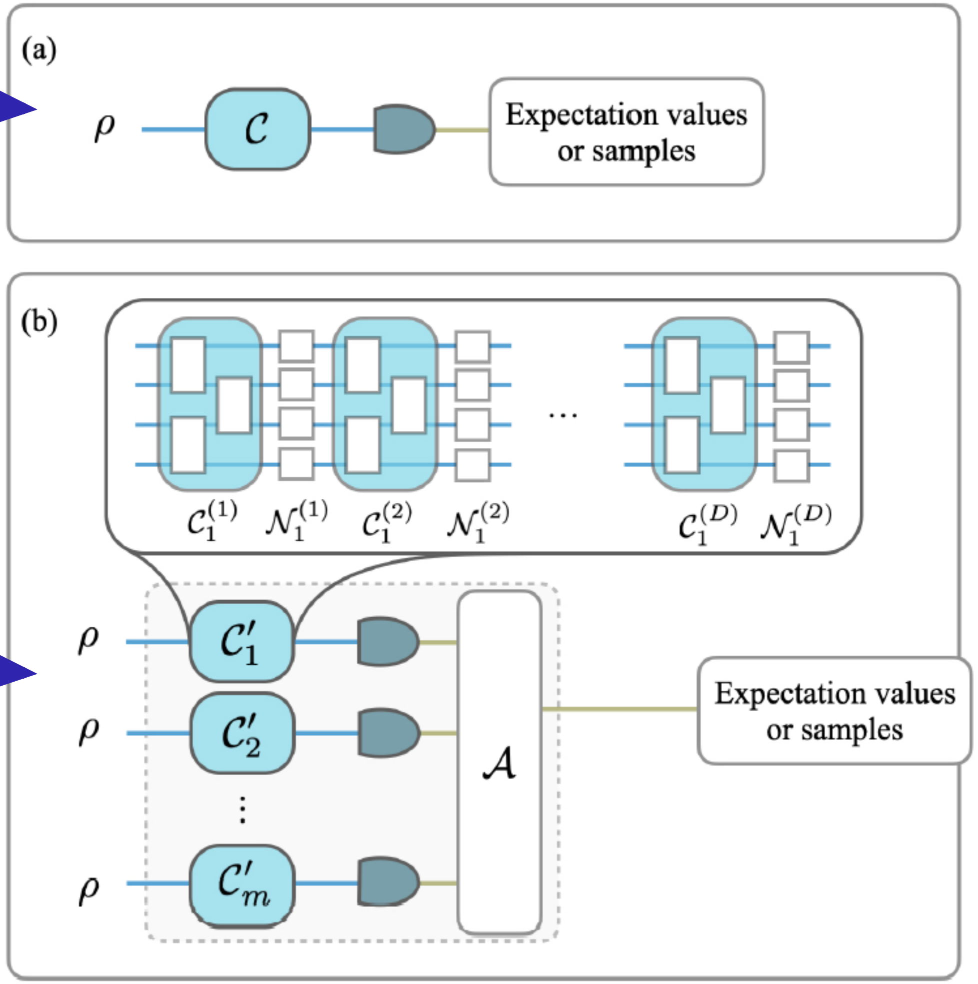
[1]Li and Benjamin, Phys. Rev. X 7, 021050 (2017). [2]Majumdar et al., IEEE Int. Conf. Quant. Comp. Eng. (2023)

Error Mitigation

IDEAL CASE
We run a target circuit on an input state and obtain the desired output

ERROR MITIGATION ABSTRACTION[1]
We run a number of noisy circuits on the input state and post-process their outputs into a prediction



[1] cf. Takagi et al., npj QI 8(1), 114 (2022); Takagi et al., PRL 131(21), 210602 (2023); Tsubouchi et al., PRL 131(21), 210601 (2023).

Limitations on Error Mitigation

LOCAL DEPOLARIZING NOISE
Pushes system state towards the maximally mixed state exponentially fast in depth[1]

WE SHOW THAT
For many circuits, this effect is drastically amplified, even in log-log depth

We need exponential-in-depth resources for quantum error mitigation[2,3]

We need exponential-in-depth and in-qubit-count resources for quantum error mitigation

$$N \geq \exp(O(D))$$

$$N \geq \exp(O(nD))$$

[1]Müller-Hermes et al., JMP 57(2) (2016). [2]Takagi et al., PRL, 131(21), 210602 (2023) [3]Tsubouchi et al., PRL 131(21), 210601 (2023).

Generic Bounds for Noisy Applications

LOCAL DEPOLARIZING NOISE
Pushes system state towards the maximally mixed state exponentially fast in depth

GENERICALITY
This holds independent of the actual unitaries in the circuit

TIGHTNESS
There exist quantum circuits that saturate this scaling

Different circuits should yield different outputs (for example expectation values)

If the states for different circuits are close to the maximally mixed state they are mutually close

If outputs are far but states are close, we have to use many copies!

Beyond Generic Bounds

DECAY IN WIDTH
does not happen generically, consider circuits that only contain single-qubit gates

WE NEED CIRCUITS
That create lots of entanglement in short depth. We construct them using unitary 2-designs[1]

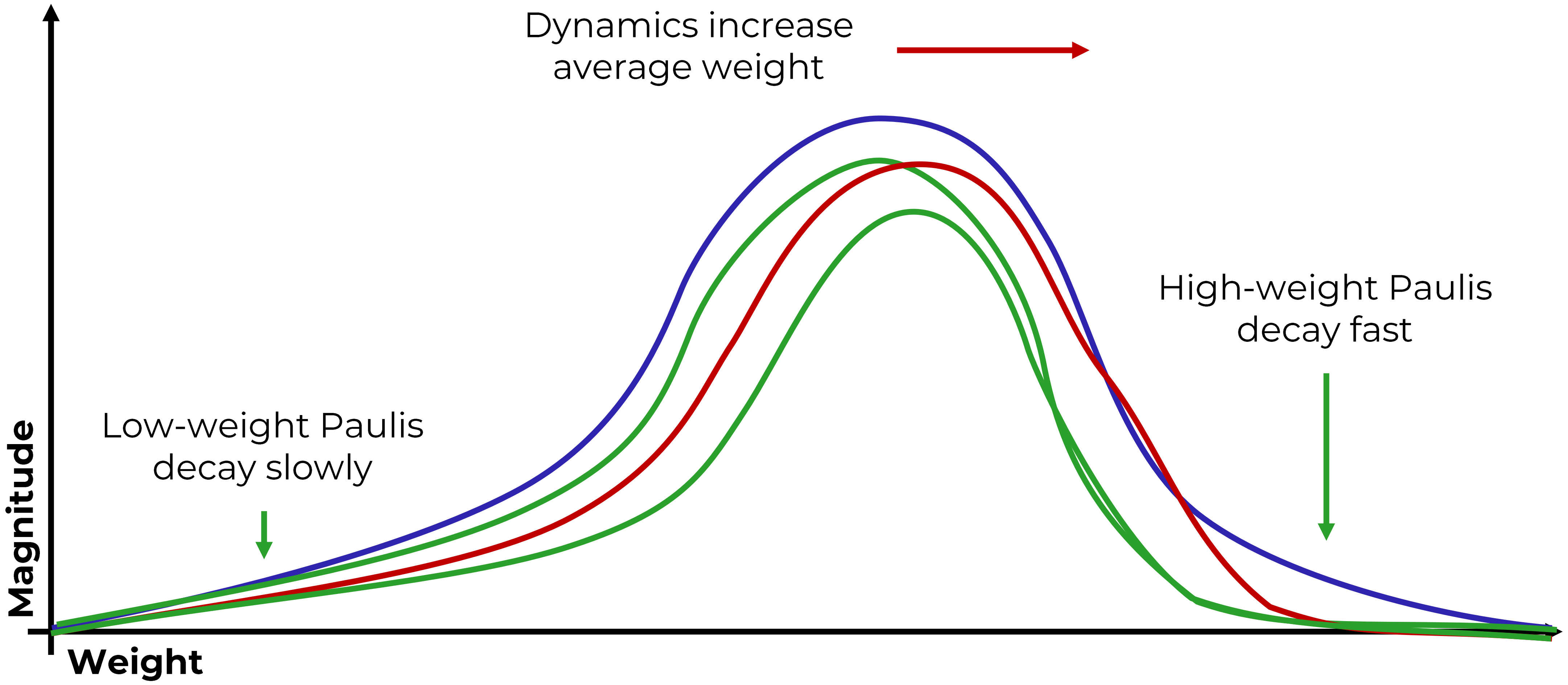
WE PROVE
That these states converge to the maximally mixed state much faster

TOOL: PAULI EXPANSION
We can expand the system state into Pauli words, there local depolarizing noise acts like

$$\mathcal{D}_\lambda[P] = \lambda^{w(P)} P$$

[1]Cleve et al., Quant. Inf. Comp. 16, 721–756 (2016)

Distribution of Pauli Coefficients



Application Takeaway

Error mitigation can still help,
but **error correction**
is what we really need

Technical Takeaway

We can prove much **stronger limitations** if we go **beyond generic arguments**

Part II

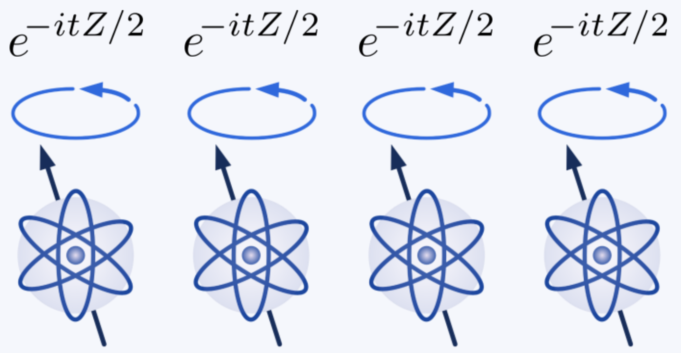
How does the picture change with **finite samples**?

Quantum Metrology

GOAL
Devise a protocol that estimates the phase as well as possible

PHASE ESTIMATION
Local evolution of an ensemble of spins under the phase Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} Z_i$$



LIMITATIONS OF PRACTICAL METROLOGY

- Low numbers of quantum systems
- Slow operation speed of certain platforms
- Drift of the system parameters

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Practical metrology happens in the **finite-sample regime!**

Traditional Quantum Metrology

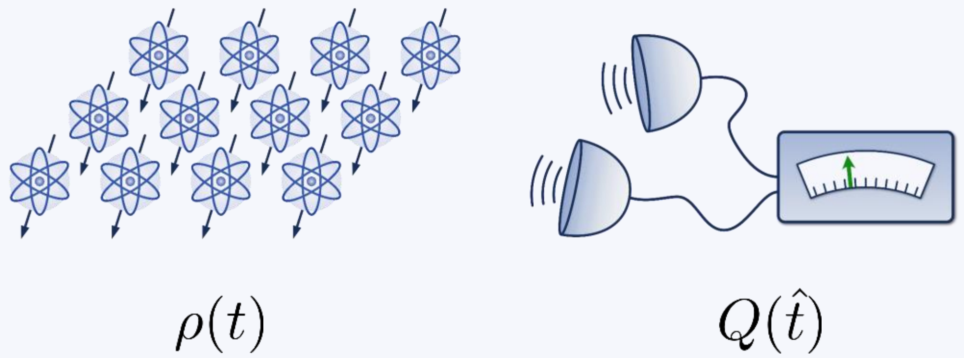


FIGURE OF MERIT

- **Mean squared error** of the estimate
- Equal to the **variance** if the estimate is exact in expectation

CRAMÉR-RAO BOUND[1]

$$\mathrm{Var}(\hat{t}) \geq \frac{1}{\mathcal{F}(t)}$$

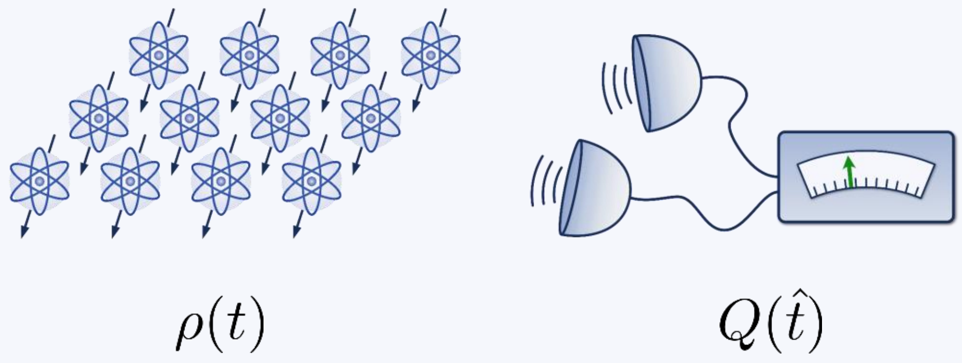
WHAT ABOUT THE FINITE-SAMPLE REGIME?

- Cramér-Rao bound still constrains precision
- But it can be **overly optimistic**

Optimizing for large Fisher information can lead to poor finite-sample performance

[1]Helstrom, *Phys. Lett. A* (1967)

Single-shot Quantum Metrology



SINGLE-SHOT FIGURE OF MERIT

- Define an **estimation tolerance** δ that should be achieved
- Determine the probability of estimating the parameter within that tolerance

SUCCESS PROBABILITY[1]

$$\mathbb{P}[|t - \hat{t}| \leq \delta]$$

[1]See also Hayashi, *J. Phys. A* (2002), Walter and Renes, *IEEE Trans. Inform. Theory* (2014)

Hypothesis Testing Bound

MOTIVATION
If states that are $O(\delta)$ apart are hard to distinguish, estimating the parameter to precision δ should also be hard

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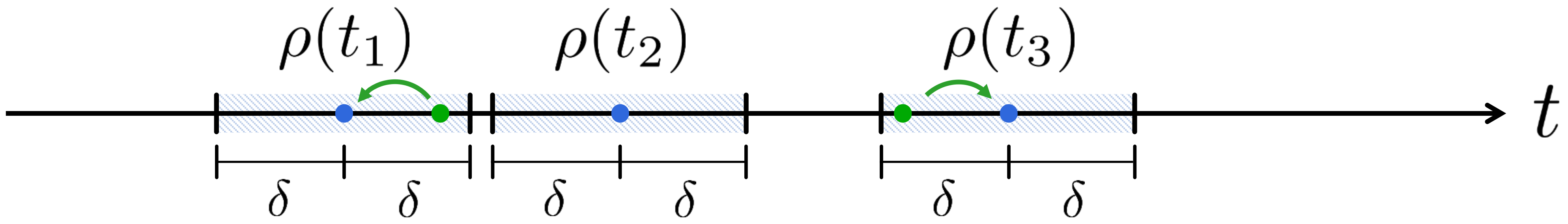
Perform a reduction from quantum metrology to multi-hypothesis testing



We can use the metrology protocol to solve the multi-hypothesis testing task

→

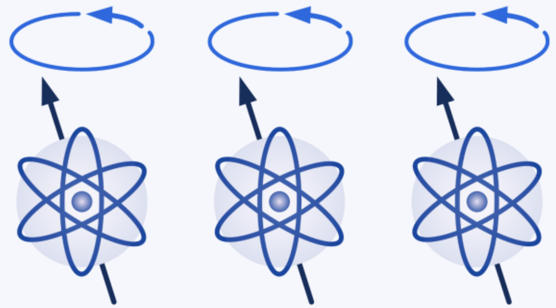
THEOREM
The success probability of quantum metrology cannot exceed the success probability of distinguishing states at times that are at least 2δ apart

Optimal Phase Estimation

PHASE ESTIMATION

Local evolution of an ensemble of spins under the phase Hamiltonian

$$e^{-itZ/2} \quad e^{-itZ/2} \quad e^{-itZ/2}$$



$$H = \frac{1}{2}\sum_{i=1}^{n} Z_i$$

We show that the **pretty good measurement**[1] is optimal for covariant state sets

↓

We use this to obtain a **closed-form solution** for the minimax success probability

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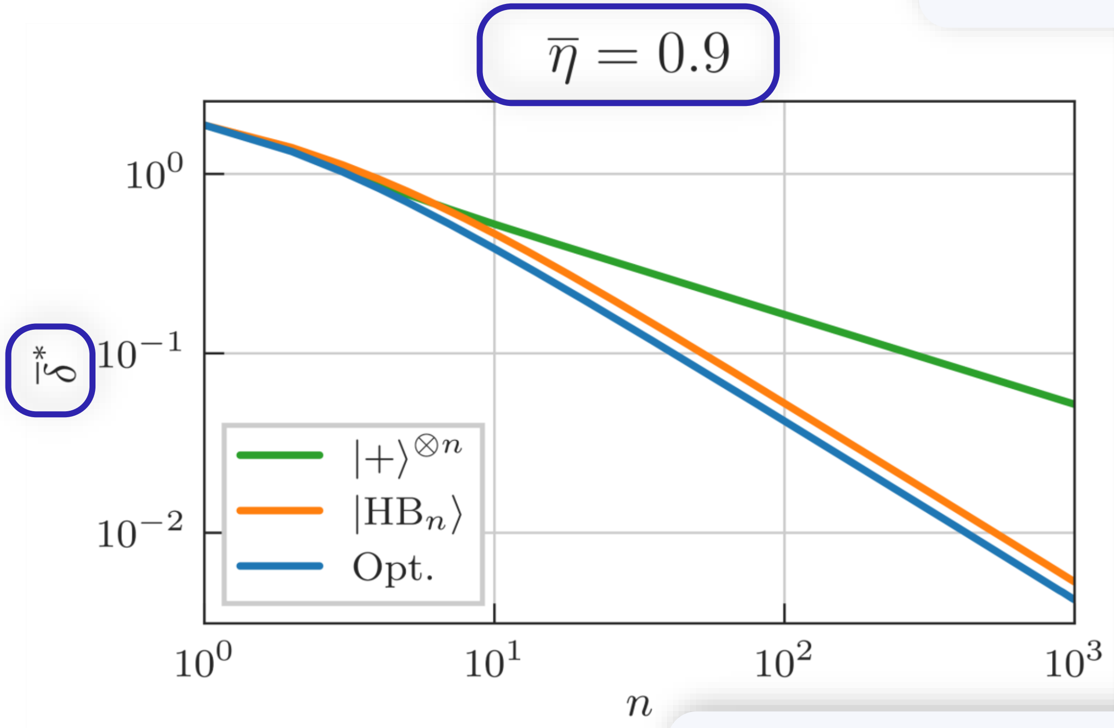
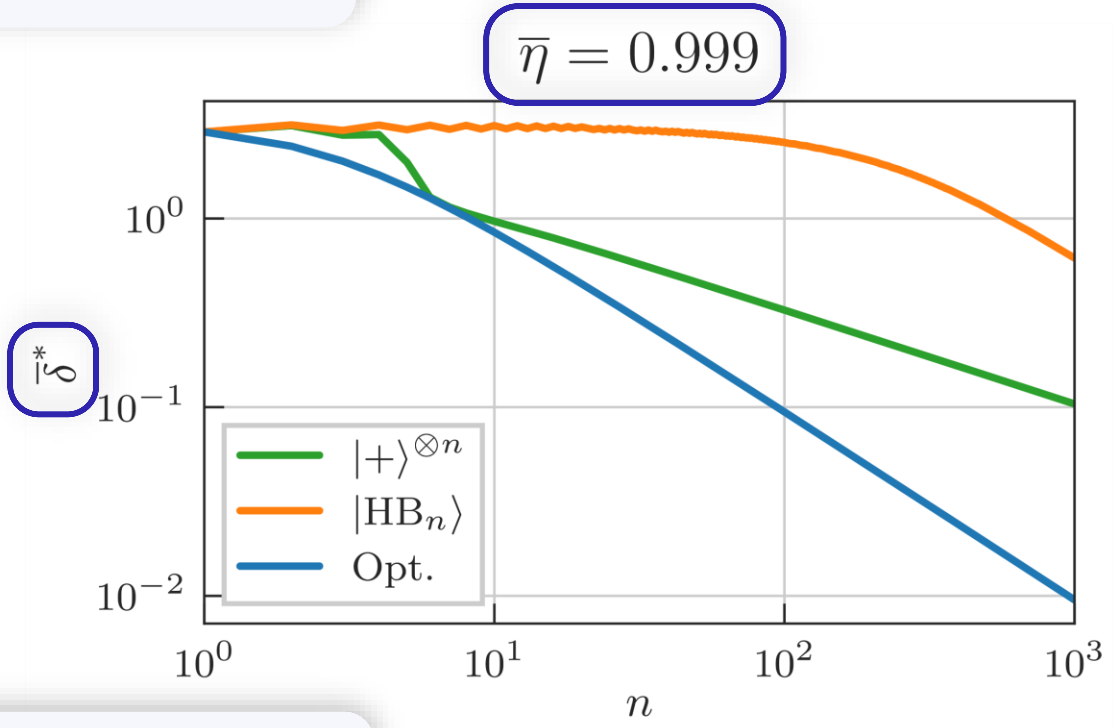
The closed-form solution factilitates a **numerical comparison** of different **probe states**

[1]Holevo, *Rep. Math. Phys.* (1997)

Minimax Estimation Tolerance

Desired probability of success





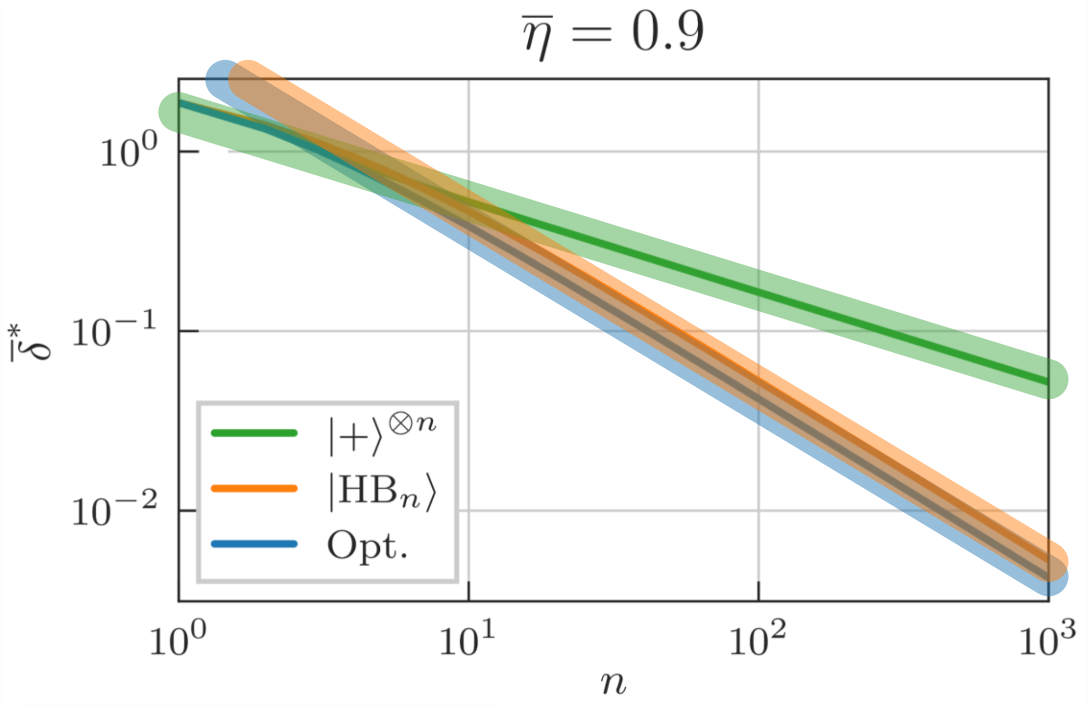
Achievable optimal tolerance
(Loglog plot)

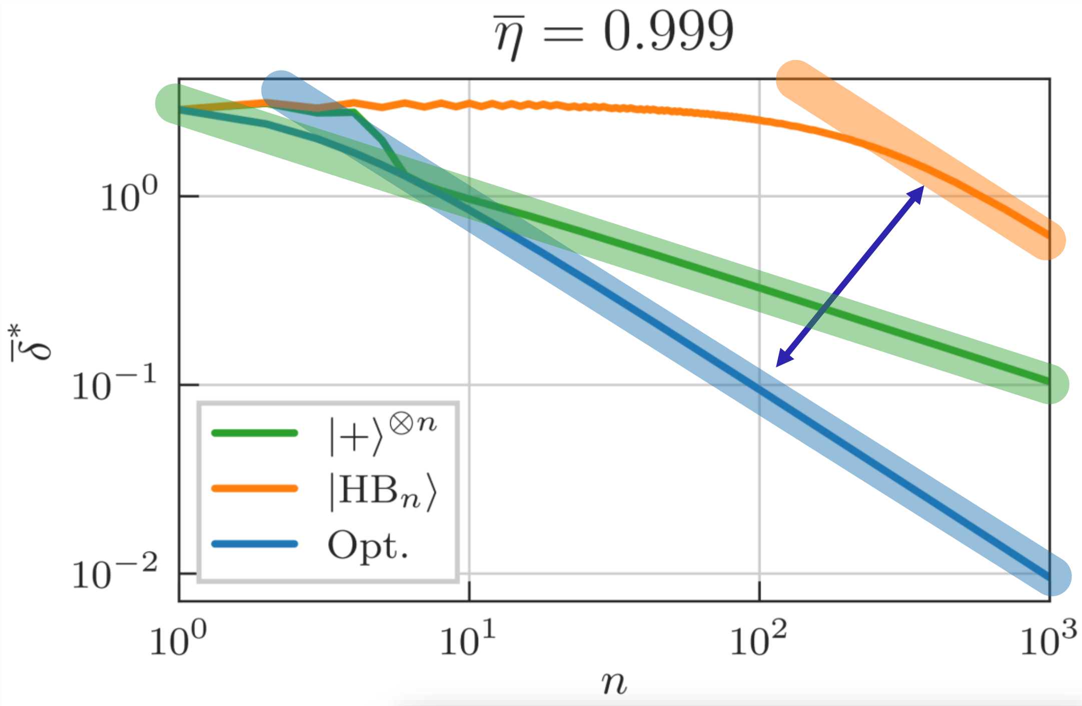
Minimax Estimation Tolerance

Holland-Burnett State[1]

$$|\mathrm{HB}_n\rangle = \frac{1}{\sqrt{n+1}}(|0\rangle + |1\rangle + |2\rangle + \cdots + |n\rangle)$$

Has similar Fisher information as optimal state





Standard Quantum Limit $\sim \frac{1}{\sqrt{n}}$

Heisenberg Limit $\sim \frac{1}{n}$

Performance gap depending on the success probability

Cannot be predicted by Fisher information!

[1]Holland and Burnett, *Phys. Rev. Lett.* (1993)

Beyond Quantum Fisher Information

The mean-squared error and the Cramér-Rao bound do not faithfully capture the finite-sample regime

↓

Quantifying the **success probability** allows for a more fine-grained analysis

↓

Allows us to rigorously show cases where the limits predicted by the Quantum Fisher Information can not be reached

Application Takeaway

Optimizing for Fisher Information may lead to **poor finite-sample performance**

Technical Takeaway

To fully understand limitations we need to **consider the** choice of **figure of merit**

Part III

How constraining are **computational restrictions**?

Reviewing our Tools

Part I

Device Noise

Sample-complexity lower bounds based on Fano‘s inequality for estimator in hypothesis testing

$$\mathbb{P}[\hat{X} \neq X] \geq 1 - \frac{I(X : \hat{X}) + \log 2}{\log M}$$

Part II

Finite Samples

Reduction to the error probability of a symmetric binary hypothesis test

$$P_e^*(\rho \,\|\, \sigma) = \frac{1}{2} - \frac{1}{4}\|\rho - \sigma\|_1$$

Reviewing our Tools

Information-theoretic tools and bounds are indispensable when studying the fundamental limitations of quantum technologies

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But these tools are often **asymptotic** in nature and assume **infinite computational power**

↓

Can we come up with information-theoretic quantities that do not impose these assumptions?

Starting Point: Relative Entropy

RELATIVE ENTROPY

$$D(\rho \,\|\, \sigma) := \mathrm{Tr}[\rho(\log \rho - \log \sigma)]$$

PARENT QUANTITY FOR

- Entropy
- Mutual Information
- Coherent Information
- Holevo Chi
- ...

Single-shot Approach to Relative Entropy

HYPOTHESIS TESTING RELATIVE ENTROPY

Operational definition of the best type II error achievable at fixed type I error

$$D_h^\epsilon(\rho \,\|\, \sigma) := -\log \min_{0 \leq \Lambda \leq \mathbb{I}} \{\, \mathrm{Tr}[\Lambda\sigma] \,|\, \mathrm{Tr}[\Lambda\rho] \geq 1 - \epsilon \}$$

QUANTUM STEIN'S LEMMA

Best asymptotic exponent in the limit of many copies

$$\lim_{n\to\infty} \frac{1}{n} D_h^\epsilon(\rho^{\otimes n} \,\|\, \sigma^{\otimes n}) = D(\rho \,\|\, \sigma)$$

Adding Complexity Restrictions

G-COMPLEXITY HYPOTHESIS TESTING RELATIVE ENTROPY[1]

Operational definition of the best type II error achievable at fixed type I error and fixed gate complexity of the measurement

$$D_h^\epsilon(\rho \,\|\, \sigma; G) := -\log \min_{0 \leq \Lambda \leq \mathbb{I}} \left\{ \mathrm{Tr}[\Lambda\sigma] \,\middle|\, \mathrm{Tr}[\Lambda\rho] \geq 1 - \epsilon, C(\Lambda) \leq G \right\}$$

COMPUTATIONAL RELATIVE ENTROPY

Regularizing this over polynomially many copies and gates gives the best exponent in the polynomial regime

$$\approx D_h^\epsilon(\rho^{\otimes \mathrm{poly}(n)} \,\|\, \sigma^{\otimes \mathrm{poly}(n)}; \mathrm{poly}(n))$$

[1]Munson et al., PRX Quantum 6 010346 (2025)

Teaser

- We define a computational relative entropy in a mathematically rigorous way
- We can show separations: there exists states that can be arbitrarily well tested without resource constraints, but not under computational constraints
- We derive a computational entropy and show that it has an interpretation as the best compression rate under computational constraints
- We show how our computational quantities can be used to understand entanglement manipulation under complexity constraints

Application Takeaway

Information-theoretic bounds can be **arbitrarily far** from the actual achievable performance

Technical Takeaway

We can define **entropic quantities** that have **computational limitations** inbuilt

Conclusion

Lots of
exciting science
is left to be done

Exciting directions

- We have a good understanding of information-theoretic limits on quantum technologies
- But in reality, these applications are very restricted and can often not saturate these limits
- Understanding the true limitations is thus still wide open, and we are still just at the beginning of fully capturing limitations that arise from
  - Limited computational power
  - Imperfect devices
  - Limited sample sizes
  - Limited capabilities (e.g. multi-copy measurements)

Thank you for your attention.

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Metrology

Mitigation

Slides