

Quantum metrology in the finite-sample regime

JOHANNES JAKOB MEYER
FU BERLIN

PERIMETER SEMINAR
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Johannes Jakob Meyer,¹ Sumeet Khatri,¹ Daniel Stilck França,^{1,2,3} Jens Eisert,^{1,4,5} and Philippe Faist¹

¹*Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany*

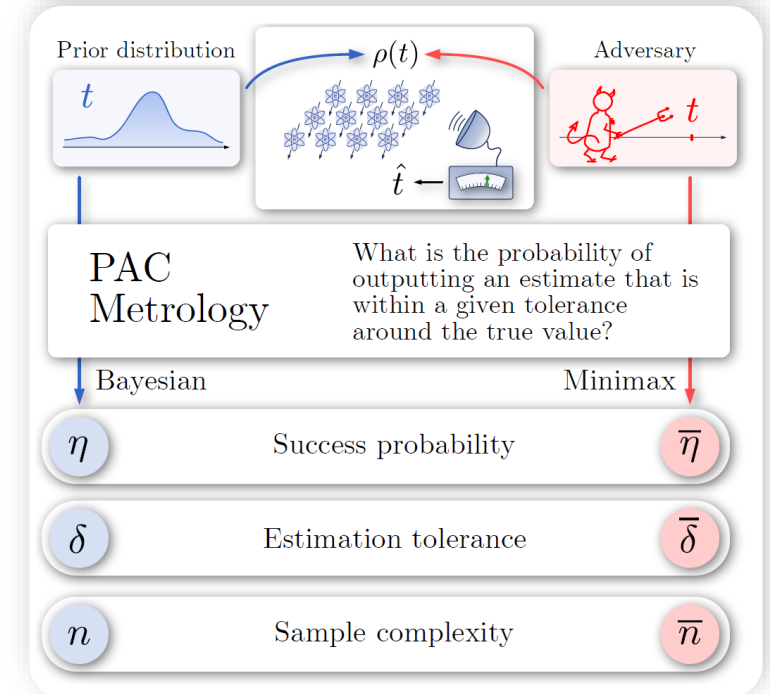
²*Department of Mathematical Sciences, University of Copenhagen, 2100 København, Denmark*

³*Ecole Normale Supérieure de Lyon, 69342 Lyon Cedex 07, France*

⁴*Helmholtz-Zentrum Berlin für Materialien und Energie, 14109 Berlin, Germany*

⁵*Fraunhofer Heinrich Hertz Institute, 10587 Berlin, Germany*

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Quantum Metrology

QUANTUM SENSING

Measure a physical parameter, e.g. a magnetic field



metrology

/mɪˈtrɒlədʒi/

noun

the scientific study of measurement.

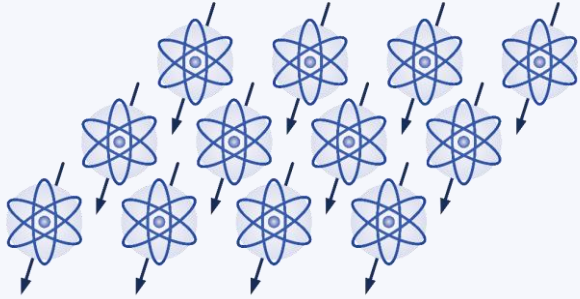
QUANTUM PARAMETER ESTIMATION

Measure a parameter encoded in a quantum system

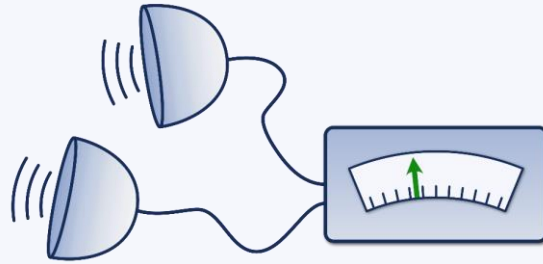
TOMOGRAPHY

Estimate the quantum state of a system

Traditional Quantum Metrology



$\rho(t)$



$Q(\hat{t})$

Want unbiased estimate

$$\mathbb{E}[\hat{t}] = t$$

with **low variance**

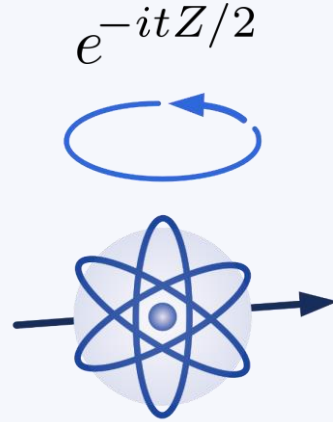
Cramér-Rao Bound

$$\text{Var}(\hat{t}) \geq \frac{1}{\mathcal{F}(t)}$$

- › Inherently asymptotic
- › Assumes parameter is already approximately known
- › Application difficult to justify in the finite-sample regime

An Example

Alice prepares a plus state, lets it evolve under a phase Hamiltonian and sends it to Bob via a channel dephasing it in the plus/minus basis



$$\rho(t) = \cos^2(t/2)|+\rangle\langle+| + \sin^2(t/2)|-\rangle\langle-|$$

$$\mathcal{F} = \begin{cases} 1 & \text{if } t \notin \{0, \pi\} \\ 0 & \text{if } t \in \{0, \pi\} \end{cases}$$

We are guaranteed there exists an observable diagonal in the plus/minus basis with

$$\langle T \rangle = t \quad \langle \Delta T \rangle = 1/\mathcal{F}$$

For small t , this observable has eigenvalues

$$T \simeq \text{diag}(O(t), O(1/t))$$

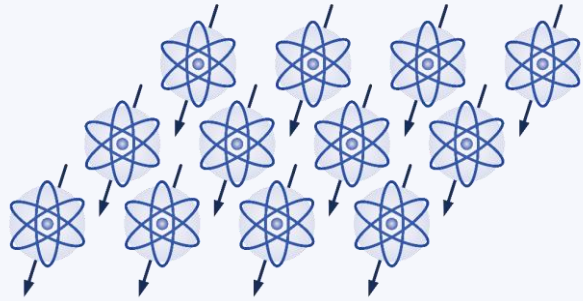
Occuring with probability

$$P \simeq (O(1), O(t^2))$$

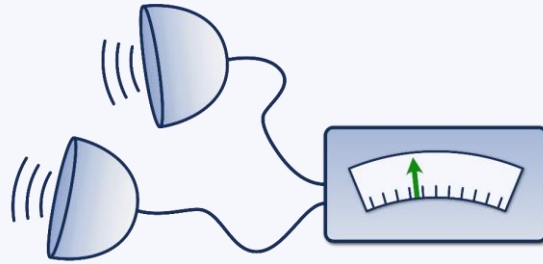
Which means resolving t takes $O(t^2)$ samples

Single-shot Quantum M

Estimation Tolerance



$\rho(t)$



$Q(\hat{t})$

$$\text{Is } |t - \hat{t}| \leq \delta?$$

Yes →

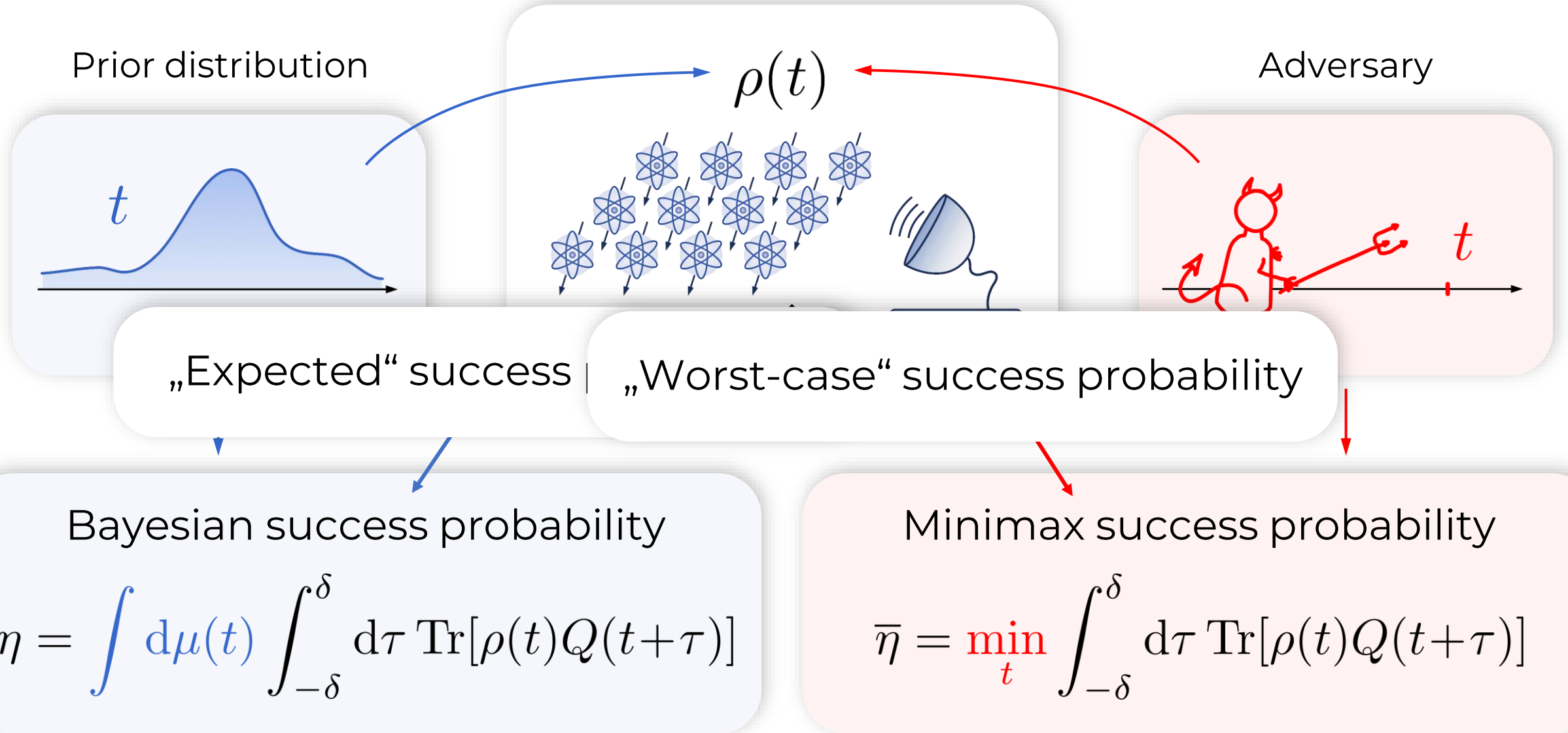
Success ✓

No →

Failure ✗

What is the probability of successful estimation?

Single-shot Quantum Metrology



The PAC Metrology Framework

$\overline{\eta}$

SUCCESS PROBABILITY

What is the probability of obtaining an estimate within a fixed tolerance?

$\overline{\delta}$

ESTIMATION TOLERANCE

What is the smallest tolerance that still guarantees a fixed success probability?


\overline{n}

SAMPLE COMPLEXITY

How many copies of a state do I need to guarantee a fixed success probability and tolerance?

Optimal Measurements

Optimal minimax success probability

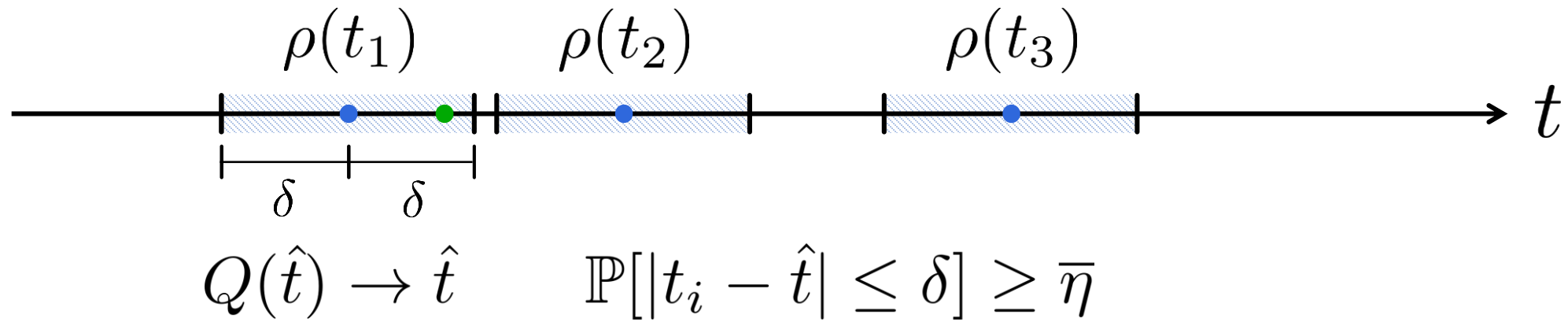
$$\bar{\eta}^* = \max_{Q(\hat{t})} \left\{ \min_t \int_{-\delta}^{\delta} d\tau \operatorname{Tr}[\rho(t)Q(t+\tau)] \right\}$$


th

- › We give a dual formulation without duality gap
- › We generalize it to the parametrized channels where we optimize over combs or strategies with indefinite causal order
- › We also give post-processing strategies for fixed measurements

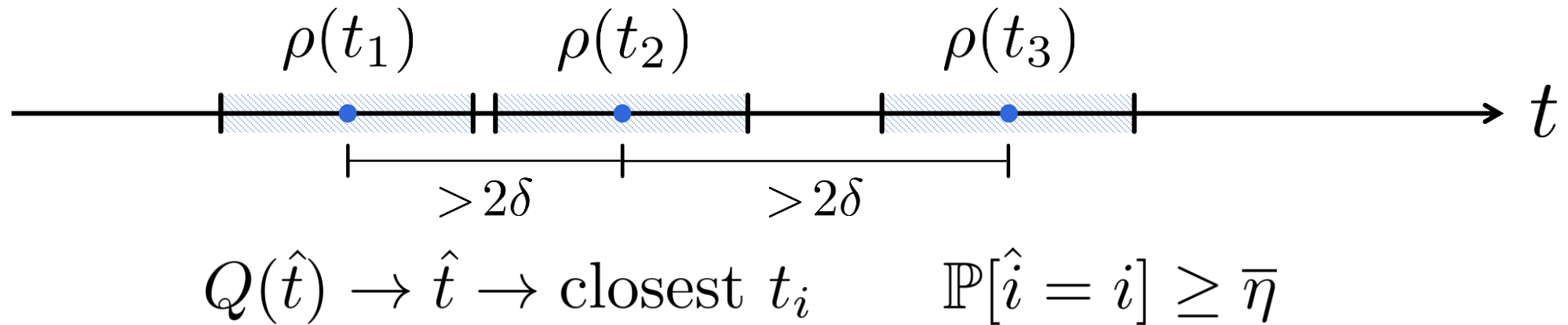
Connection to Hypothesis Testing

Metrology problem



Connection to Hypothesis Testing

Multi-hypothesis testing problem



We conclude that

$$\bar{\eta} \leq \overline{P}_s(\{\rho(t_i)\}) \text{ as long as } |t_i - t_j| > 2\delta$$

Estimation Tolerance

So far, we analyzed the success probability at fixed tolerance. But in applications, we often care about the achievable precision at fixed success probability.

Minimax estimation tolerance

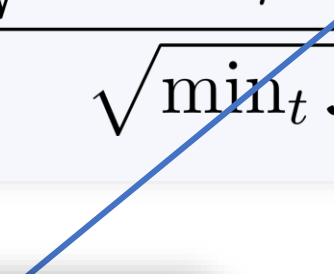
$$\bar{\delta}(\bar{\eta}) = \inf \left\{ \delta' \geq 0 \mid \bar{\eta} \leq \min_t \int_{-\delta'}^{\delta'} d\tau \operatorname{Tr}[\rho(t)Q(t+\tau)] \right\}$$

Finite-sample Cramér-Rao bound

Cramér-Rao bound

$$\sigma(\hat{t}) \geq \frac{1}{\sqrt{\mathcal{F}(t)}}$$

Our bound

$$\bar{\delta} \geq \frac{O\left(\sqrt{\log \frac{1}{1-\bar{\eta}}} - \boxed{q} \log \frac{1}{1-\bar{\eta}}\right)}{\sqrt{\min_t \mathcal{F}(t)}}$$


In the i.i.d. case

$$q = O\left(\frac{1}{\sqrt{n}}\right)$$

Sample Complexity

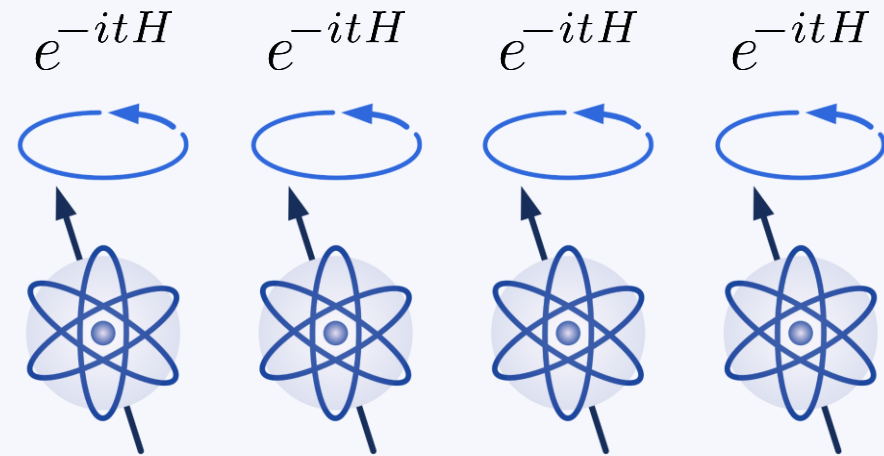
What if we care about both the achievable precision and the success probability? Then we have to ask how many copies of a state we need to achieve it.

Minimax sample complexity

$$\bar{n}(\bar{\eta}, \bar{\delta}) = \min \left\{ n' \in \mathbb{N} \mid \bar{\eta} \leq \min_t \int_{-\bar{\delta}}^{\bar{\delta}} d\tau \operatorname{Tr}[\rho^{\otimes n'}(t) Q_{n'}(t+\tau)] \right\}$$

Phase estimation

Local evolution of
an ensemble of spins under
the same phase Hamiltonian



For the regular phase Hamiltonian and $t \in [0, 2\pi)$
this yields a **covariant** set of states

Optimal Measurement

We show that the **pretty good measurement** is optimal for covariant state sets

We use this result to obtain a closed-form solution for the minimax success probability

$$\bar{\eta}^*(\delta, \psi) = \sum_{\lambda, \lambda'} |\psi_\lambda| |\psi_{\lambda'}| \frac{\sin(\delta(\lambda - \lambda'))}{\pi(\lambda - \lambda')}$$

$$H = \sum_{\lambda} \lambda \Pi_{\lambda}$$
$$|\psi\rangle = \sum_{\lambda} \Pi_{\lambda} |\psi\rangle = \sum_{\lambda} \psi_{\lambda} |\psi_{\lambda}\rangle$$

Comparison of Probe States

The closed-form solution facilitates a numerical comparison of different probe states

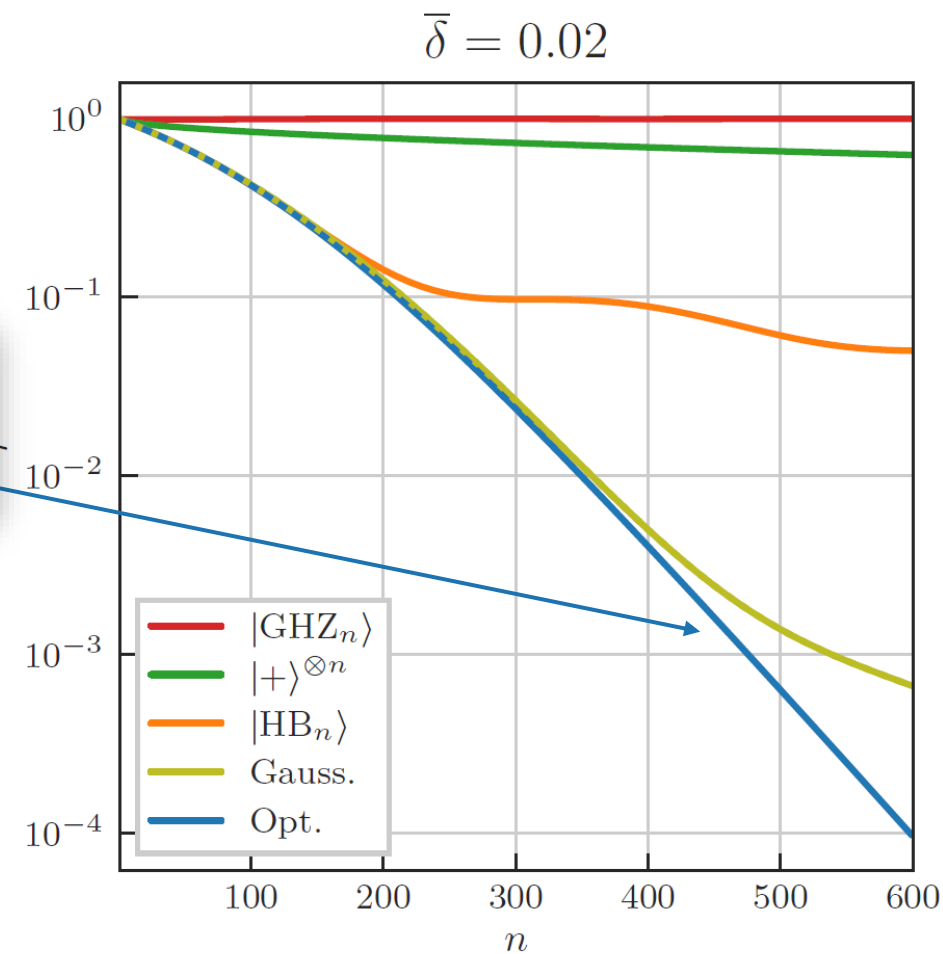
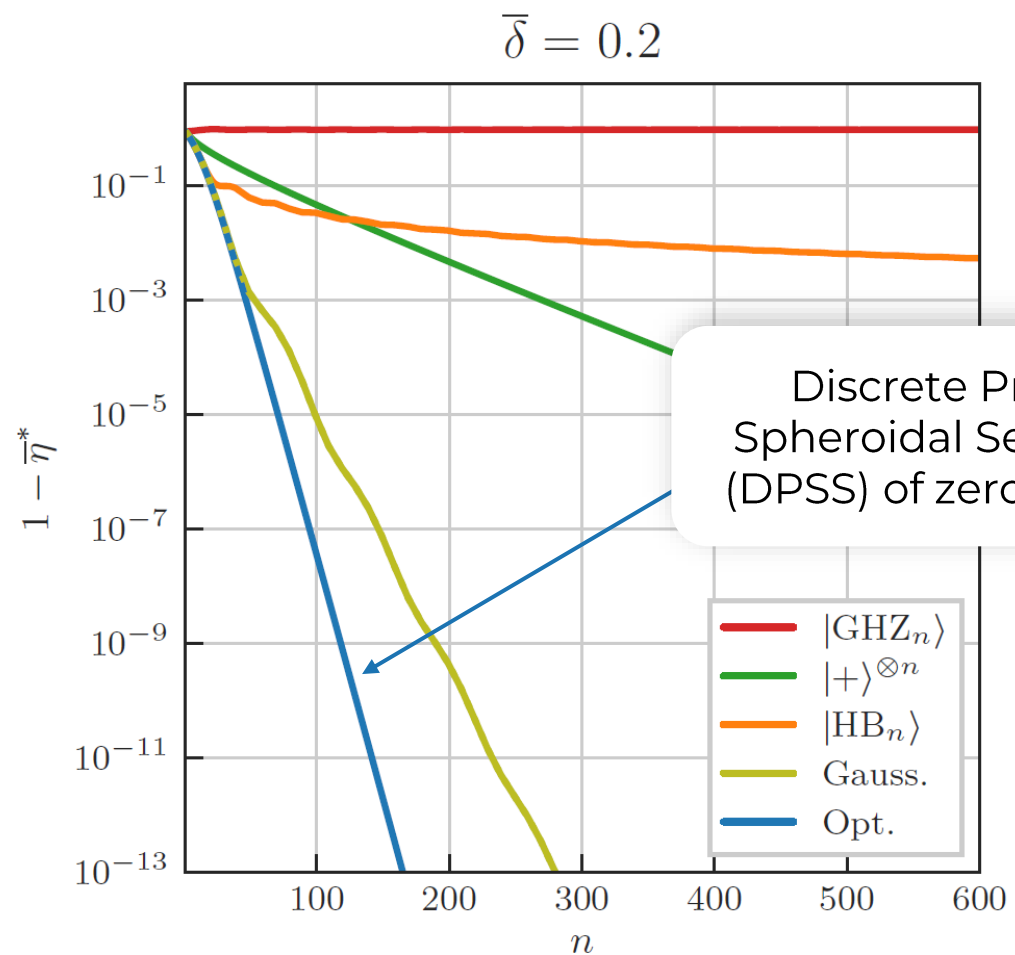
GHZ

$$|\text{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |n\rangle)$$

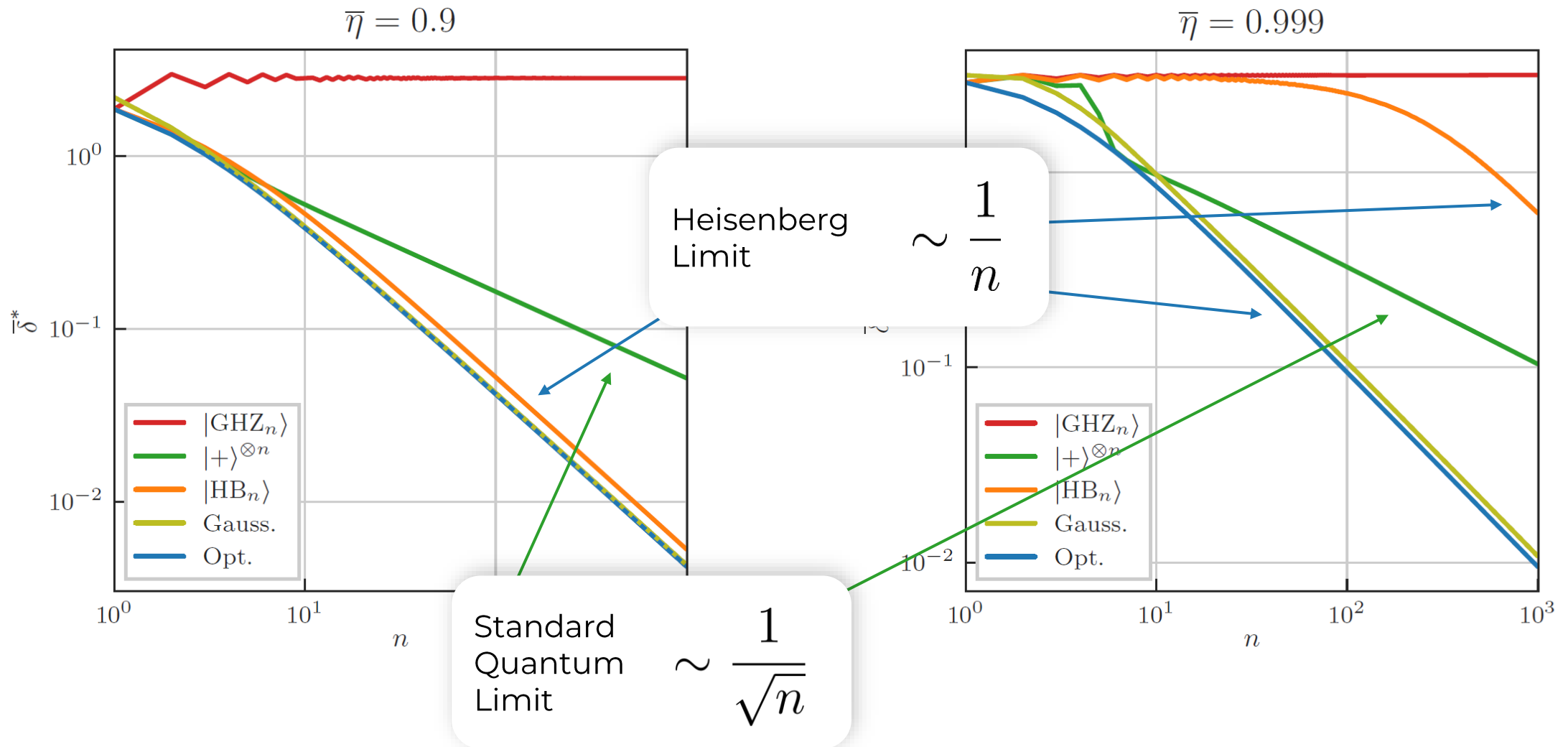
Holland-Burnett

$$|\text{HB}_n\rangle = \frac{1}{\sqrt{n+1}}(|0\rangle + |1\rangle + |2\rangle + \cdots + |n\rangle)$$

Success Probability

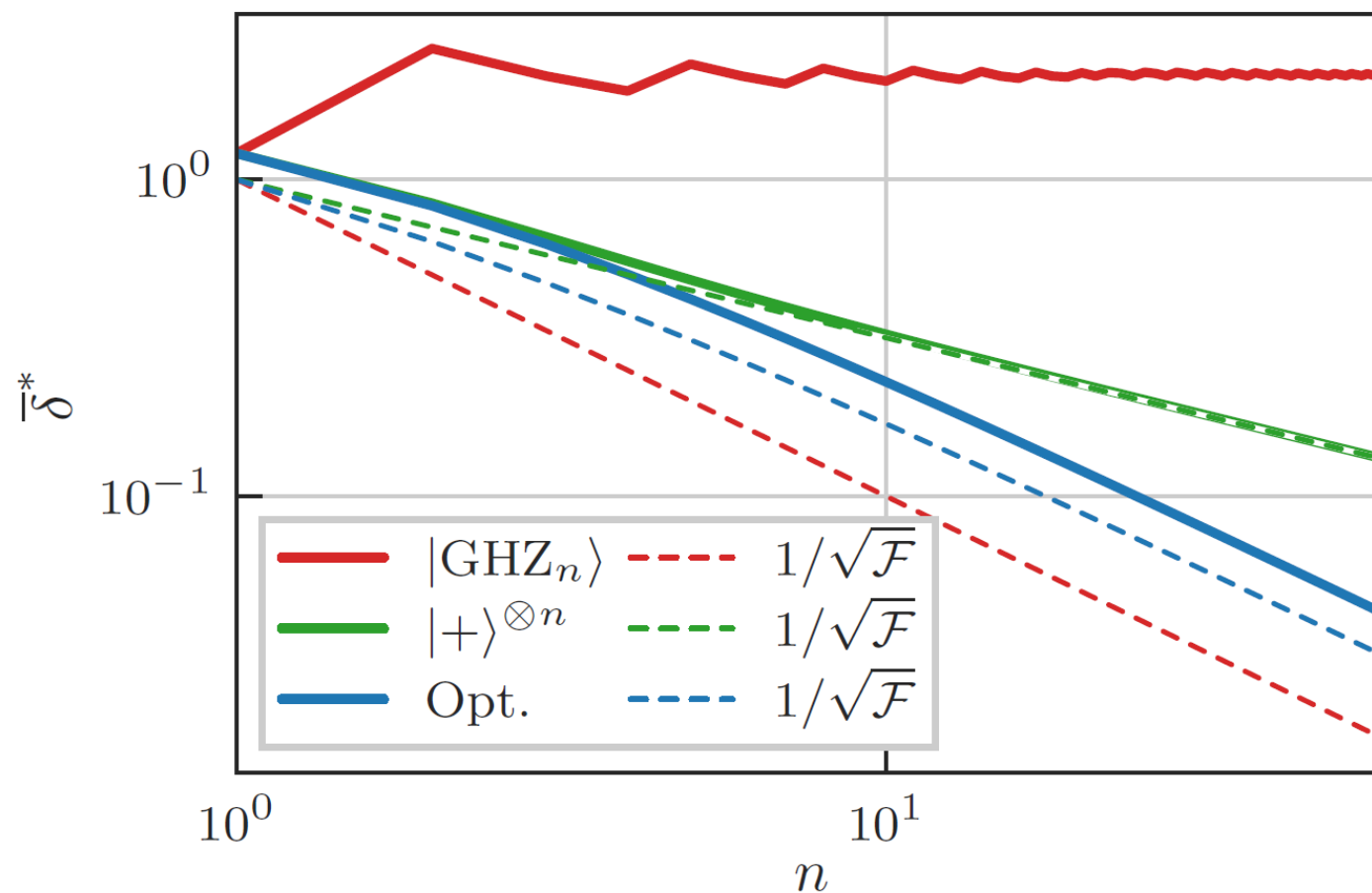


Estimation Tolerance



Comparison with QCRB

$$\bar{\eta} = \text{erf}(1/\sqrt{2})$$



Further Results in the Paper

- › We connect our quantities to single-shot entropy measures
- › We lift the hypothesis testing connection to quantum channels with different access models
- › We discuss many possible extensions of our results and definitions, e.g. the multi-parameter case
- › We give an overview of open questions

Open Questions

- › What measurements (i.e. POVMs) give good out-of-the-box performance guarantees? Pretty good measurement?
- › Improved finite-sample analogues of the Cramér-Rao bound
- › Understanding the advantages of adaptive processing and entanglement
- › What are the admissible scalings with mixed asymptotics?
- › How do noise and error correction fit into this picture?

Summary

- › We give new tools to understand quantum metrology in the single-shot regime
- › Our framework is very close to quantum information theory both in tools as in results
- › A plethora of open questions ranging from practically oriented to completely information-theoretic
- › An exciting opportunity to explore new directions in quantum metrology!

Thank you for your attention!



Slides



arXiv:2307.06370