Quantum metrology in the finite-sample regime

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Quantum metrology in the finite-sample regime

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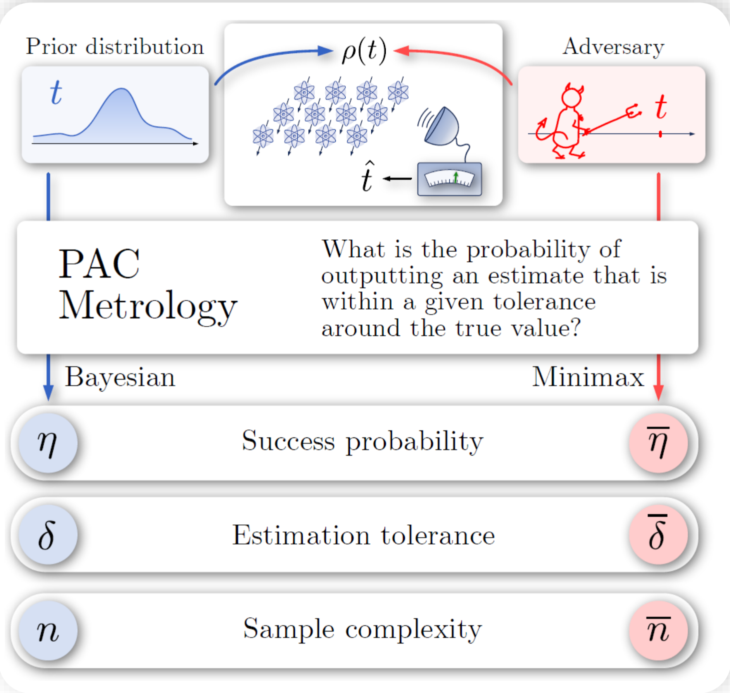
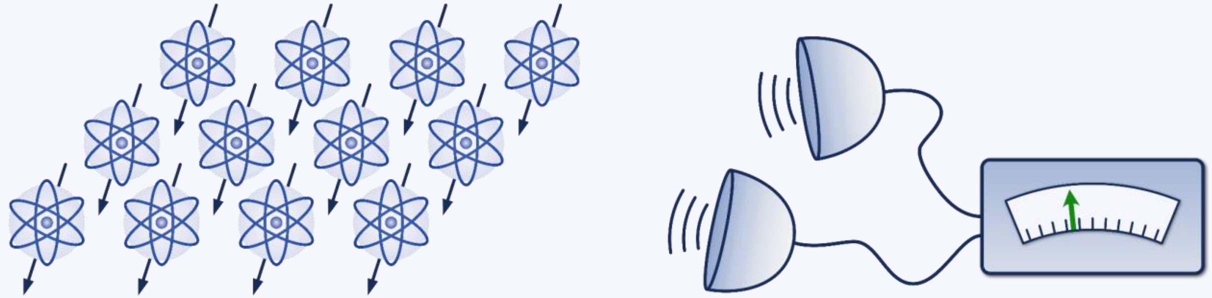


Traditional Quantum Metrology



$\rho(t)$ $\qquad$ $Q(\hat{t})$

Want unbiased estimate

$$\mathbb{E}[\hat{t}] = t$$

with **low variance**

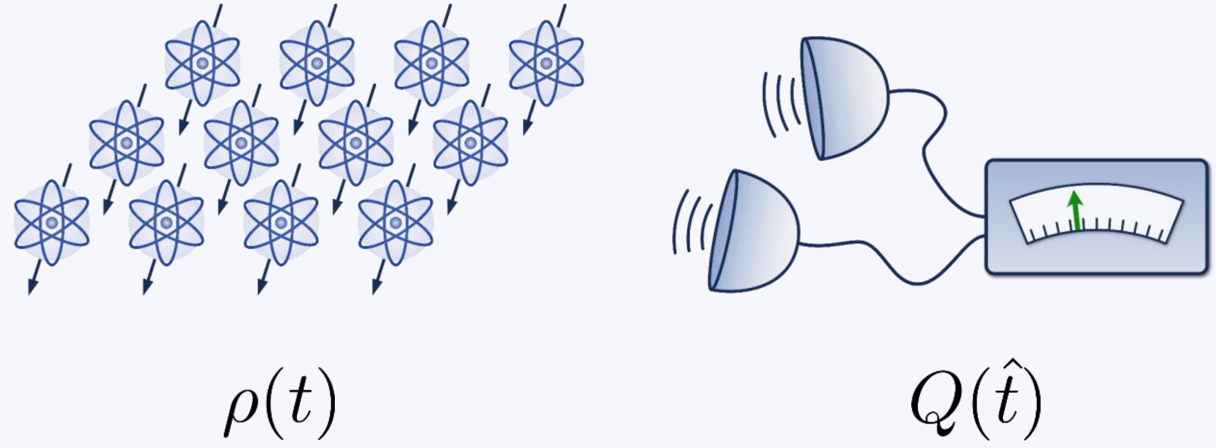
Cramér-Rao Bound

$$\mathrm{Var}(\hat{t}) \geq \frac{1}{\mathcal{F}(t)}$$

- Inherently asymptotic
- Assumes parameter is already approximately known
- Application difficult to justify in the finite-sample regime

Single-shot Quantum M

Estimation Tolerance



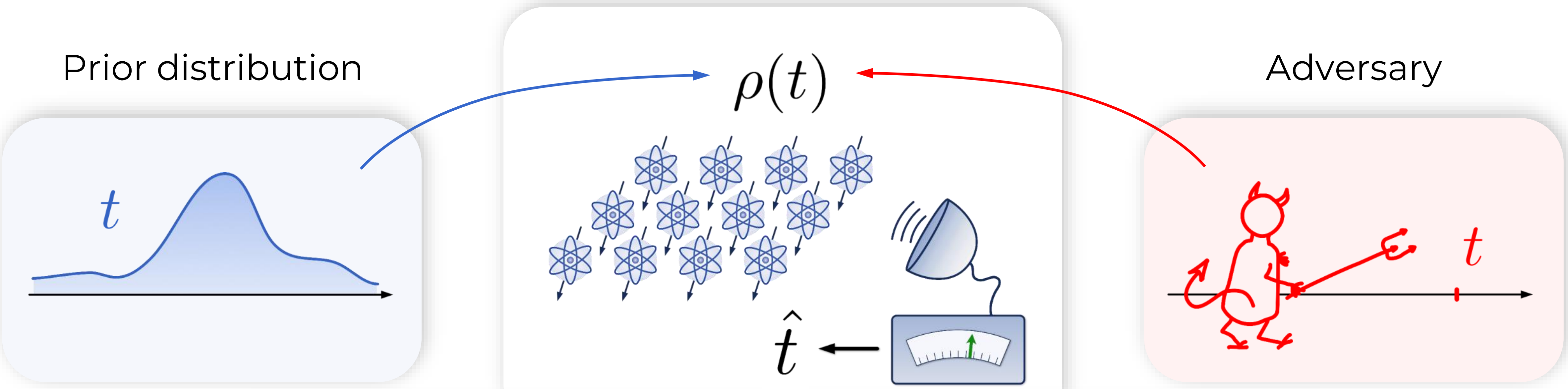
Is $|t - \hat{t}| \leq \boxed{\delta}$?

Yes → Success ✓

No → Failure ✗

What is the probability of successful estimation?

Single-shot Quantum Metrology



„Expected“ success probability

„Worst-case“ success probability

Bayesian success probability

$$\eta = \int \mathrm{d}\mu(t) \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)]$$

Minimax success probability

$$\overline{\eta} = \min_{t} \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)]$$

The PAC Metrology Framework

$\overline{\eta}$

SUCCESS PROBABILITY

What is the probability of obtaining an estimate within a fixed tolerance?

$\overline{\delta}$

ESTIMATION TOLERANCE

What is the smallest tolerance that still guarantees a fixed success probability?

$\overline{n}$

SAMPLE COMPLEXITY

How many copies of a state do I need to guarantee a fixed success probability and tolerance?

Optimal Measurements

Optimal minimax success probability

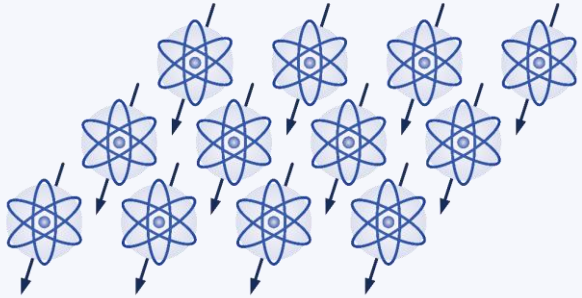
$$\overline{\eta}^* = \boxed{\max_{Q(\hat{t})}} \left\{ \min_t \int_{-\delta}^{\delta} \mathrm{d}\tau \, \mathrm{Tr}[\rho(t) Q(t+\tau)] \right\}$$

Constitutes a **semi-infinite program**,
think a continuous semi-definite program

- We give a dual formulation without duality gap
- We generalize it to the parametrized channels where we optimize over combs or strategies with indefinite causal order
- We also give post-processing strategies for fixed measurements

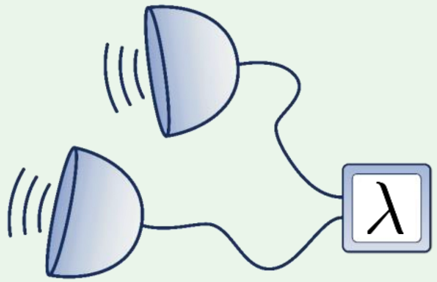
Fixed Measurements

Parametrized state



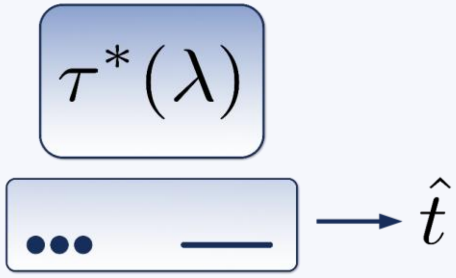
$\rho(t)$

Fixed measurement



$M(\lambda)$

Post-processing



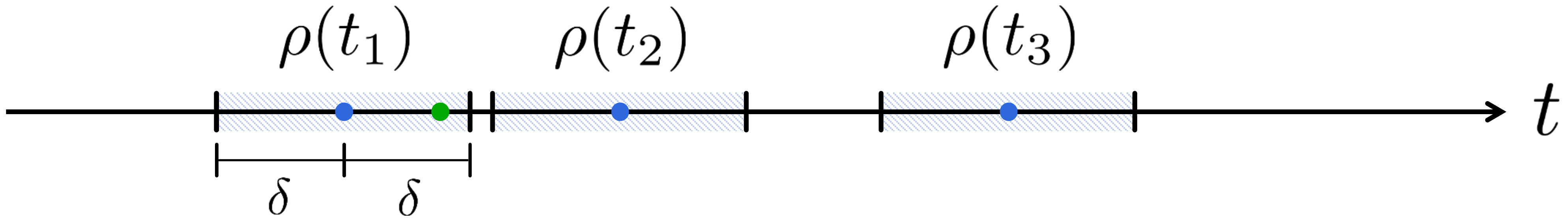
$\hat{t} = \tau^*(\lambda)$

Optimal post-processing is given by the
smoothed maximum a-posteriori estimator

$$\tau^*_{\mathrm{SMAP}}(\lambda) = \underset{t}{\operatorname{argmax}} \int_{t-\delta}^{t+\delta} \mathrm{d}\mu(\tau)\ \mathrm{Tr}[\rho(\tau) M(\lambda)]$$

Connection to Hypothesis Testing

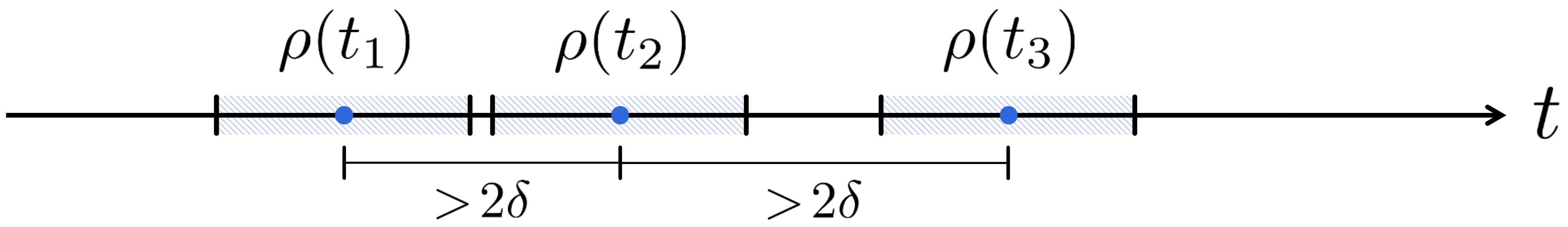
Metrology problem



$$Q(\hat{t}) \to \hat{t} \qquad \mathbb{P}[|t_i - \hat{t}| \leq \delta] \geq \overline{\eta}$$

Connection to Hypothesis Testing

Multi-hypothesis testing problem



$Q(\hat{t}) \to \hat{t} \to \text{closest } t_i$ $\quad$ $\mathbb{P}[\hat{i} = i] \geq \overline{\eta}$

We conclude that

$$\overline{\eta} \leq \overline{P}_s(\{\rho(t_i)\}) \text{ as long as } |t_i - t_j| > 2\delta$$

Estimation Tolerance

So far, we analyzed the success probability at fixed tolerance. But in applications, we often care about the achievable precision at fixed success probability.

Minimax estimation tolerance

$$\bar{\delta}(\bar{\eta}) = \inf \left\{ \delta' \geq 0 \;\middle|\; \bar{\eta} \leq \min_t \int_{-\delta'}^{\delta'} \mathrm{d}\tau \; \mathrm{Tr}[\rho(t) Q(t+\tau)] \right\}$$

Finite-sample Cramér-Rao bound

Cramér-Rao bound

$$\sigma(\hat{t}) \geq \frac{1}{\sqrt{\mathcal{F}(t)}}$$

Our bound

$$\overline{\delta} \geq \frac{O\left(\sqrt{\log \frac{1}{1-\overline{\eta}}} - \boxed{q} \log \frac{1}{1-\overline{\eta}}\right)}{\sqrt{\min_t \mathcal{F}(t)}}$$

In the i.i.d. case

$$q = O\left(\frac{1}{\sqrt{n}}\right)$$

Sample Complexity

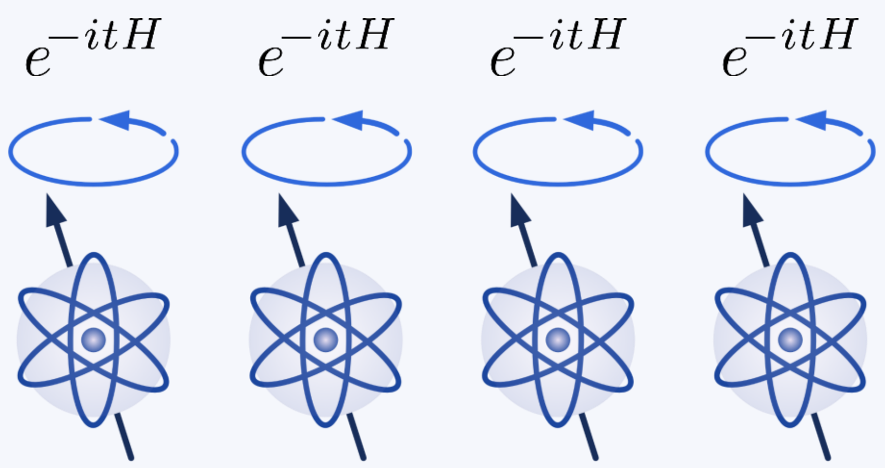
What if we care about both the achievable precision and the success probability? Then we have to ask how many copies of a state we need to achieve it.

Minimax sample complexity

$$\overline{n}(\overline{\eta}, \overline{\delta}) = \min \left\{ n' \in \mathbb{N} \;\middle|\; \overline{\eta} \leq \min_{t} \int_{-\overline{\delta}}^{\overline{\delta}} \mathrm{d}\tau \ \mathrm{Tr}[\rho^{\otimes n'}(t) Q_{n'}(t+\tau)] \right\}$$

Phase estimation

Local evolution of
an ensemble of spins under
the same phase Hamiltonian



For the regular phase Hamiltonian and $t \in [0, 2\pi)$
this yields a **covariant** set of states

Optimal Measurement

We show that the **pretty good measurement** is optimal for covariant state sets

We use this result to obtain a closed-form solution for the minimax success probability

$$\overline{\eta}^*(\delta, \psi) = \sum_{\lambda,\lambda'} |\psi_\lambda||\psi_{\lambda'}| \frac{\sin(\delta(\lambda - \lambda'))}{\pi(\lambda - \lambda')}$$

$$H = \sum_\lambda \lambda \Pi_\lambda$$

$$|\psi\rangle = \sum_\lambda \Pi_\lambda |\psi\rangle = \sum_\lambda \psi_\lambda |\psi_\lambda\rangle$$

Comparison of Probe States

The closed-form solution factilitates a numerical comparison of different probe states

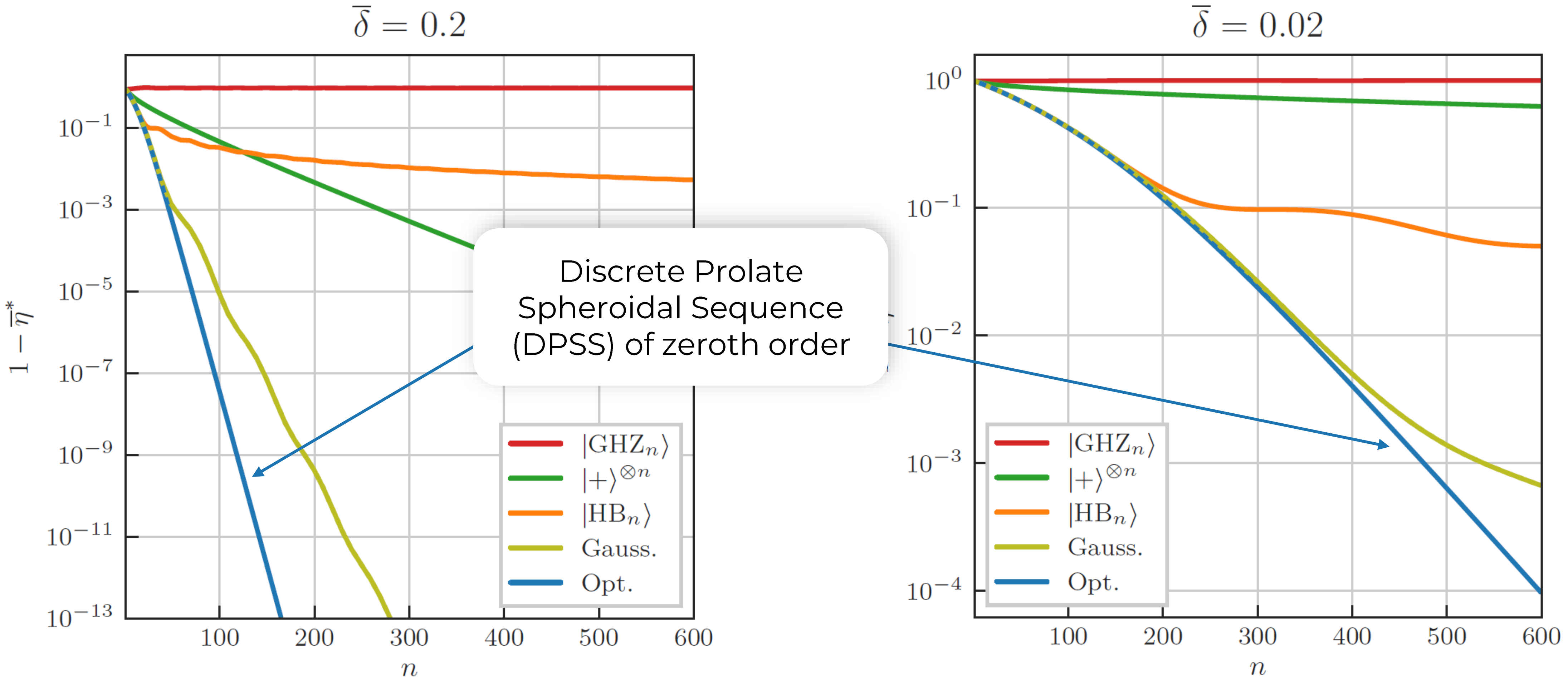
GHZ

$$|\mathrm{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |n\rangle)$$

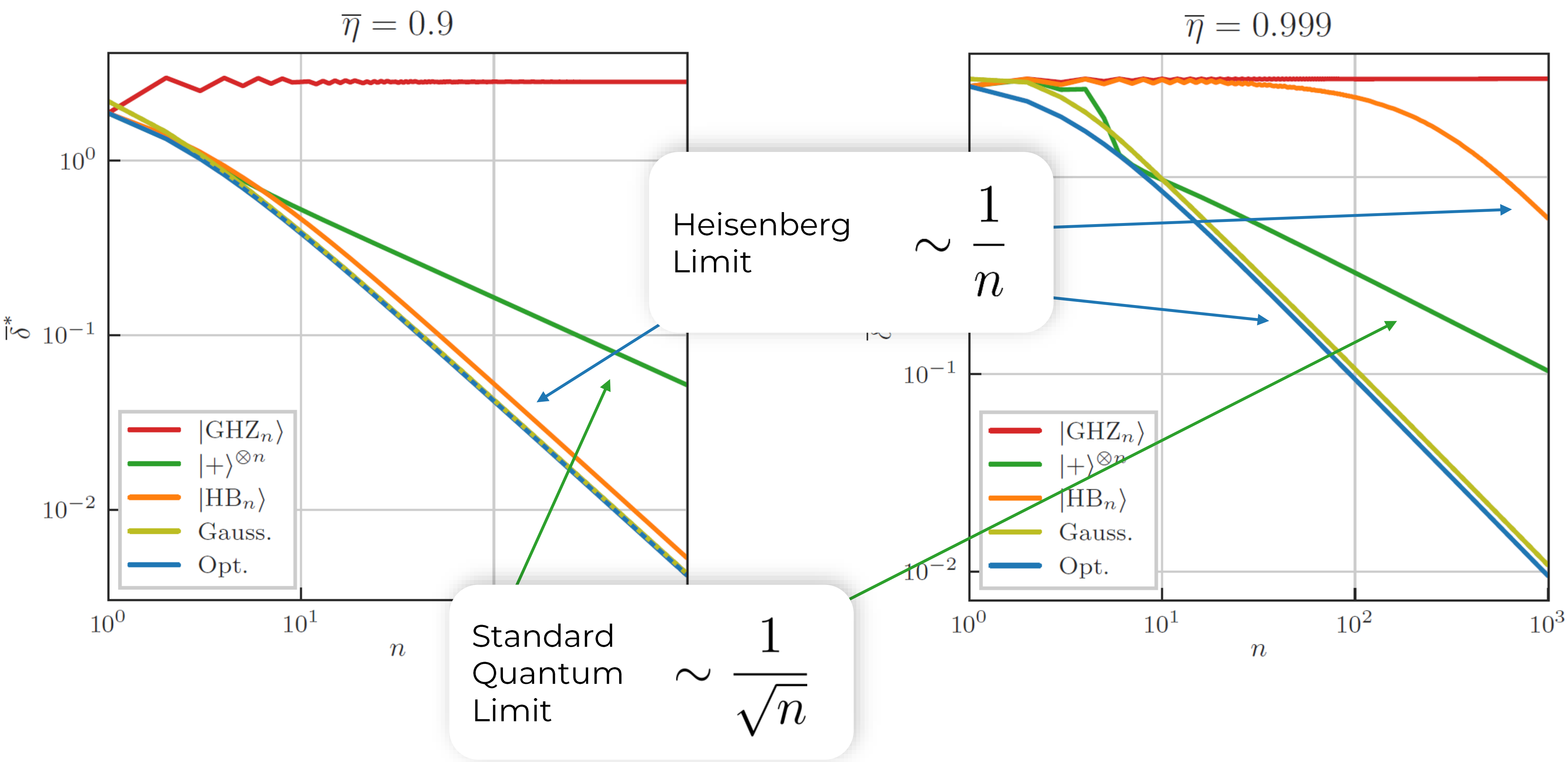
Holland-Burnett

$$|\mathrm{HB}_n\rangle = \frac{1}{\sqrt{n+1}}(|0\rangle + 1\rangle + |2\rangle + \cdots + |n\rangle)$$

Success Probability



Estimation Tolerance



Further Results in the Paper

- We connect our quantities to single-shot entropy measures
- We lift the hypothesis testing connection to quantum channels with different access models
- We discuss many possible extensions of our results and definitions, e.g. the multi-parameter case
- We give an overview of open questions

Open Questions

- What measurements (i.e. POVMs) give good out-of-the-box performance guarantees? Pretty good measurement?
- Improved finite-sample analogues of the Cramér-Rao bound
- Understanding the advantages of adaptive processing and entanglement
- What are the admissible scalings with mixed asymptotics?

Summary

- We give new tools to understand quantum metrology in the single-shot regime
- Our framework is very close to quantum information theory both in tools as in results
- A plethora of open questions ranging from practically oriented to completely information-theoretic
- An exciting opportunity to explore new directions in quantum metrology!

Thank you for your attention!

Slides

arXiv:2307.06370

Asymptotics

Asymptotic rate at constant tolerance

$$\overline{R}(\delta,\rho) := \lim_{n\to\infty} -\frac{1}{n}\log\left(1-\overline{\eta}(\delta,\rho^{\otimes n})\right)$$

Hypothesis testing bound implies

$$\overline{R}(\delta,\rho) \leq \inf_{|t-t'|>2\delta} C(\rho(t),\rho(t'))$$

We give the following achievable lower bound

$$\overline{R}(\delta,\rho) \geq \sup_{\{\mathcal{M}^{(n)}\}_{n\in\mathbb{N}}} \inf_{|t-t'|>2\delta} \overline{R}(\rho(t),\rho(t'),\{\mathcal{M}^{(n)}\}_{n\in\mathbb{N}})$$

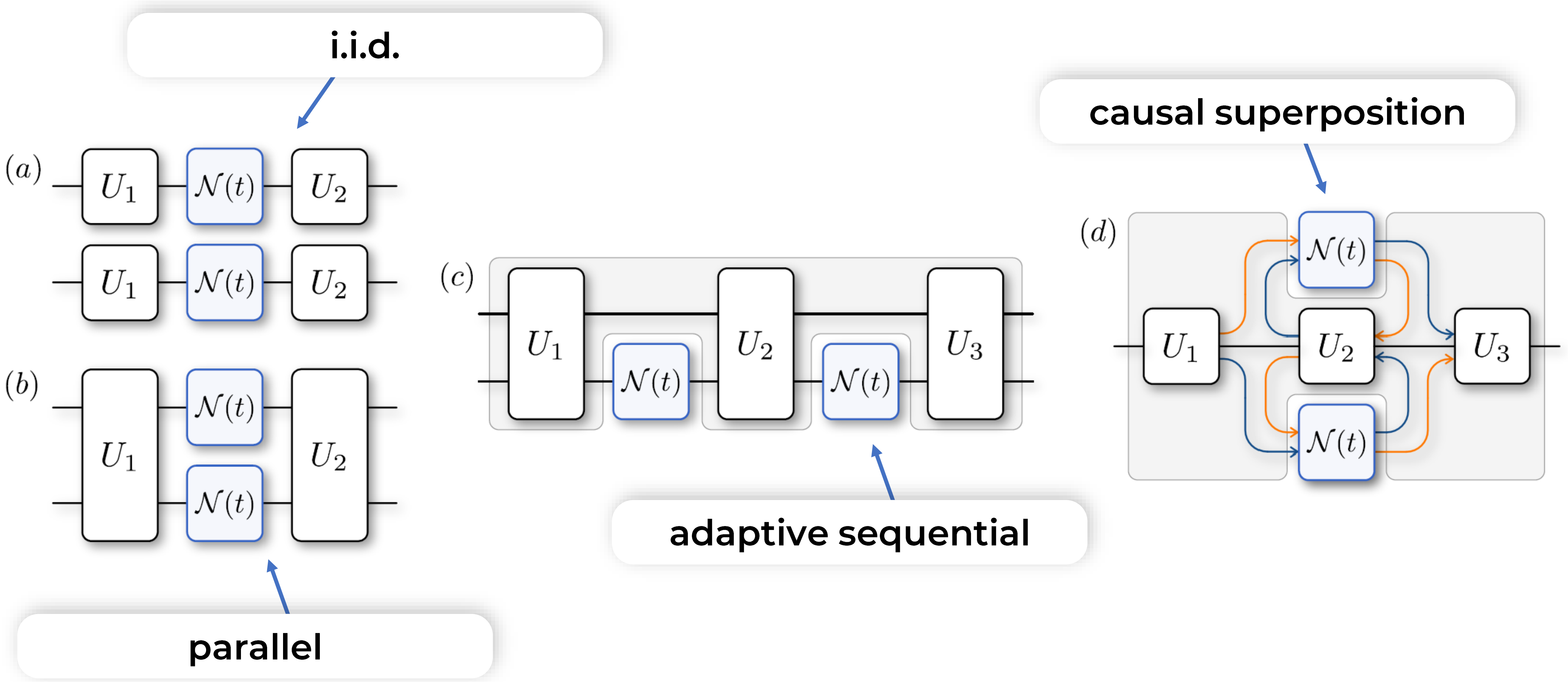
This allows us to compu

$$\overline{R}(\delta,\rho) = \inf_{|t-t'|>2\delta}$$

Hypothesis testing rate for a given measurement sequence

$$\overline{R}(\rho,\sigma,\{\mathcal{M}^{(n)}\}_{n\in\mathbb{N}}) := \lim_{n\to\infty} -\frac{1}{n}\log\left(\overline{P}_e(\mathcal{M}^{(n)}[\rho^{\otimes n}],\mathcal{M}^{(n)}[\sigma^{\otimes n}]\right)$$

Access Modes for Channels



Comparison with QCRB

$$\bar{\eta} = \mathrm{erf}(1/\sqrt{2})$$