DPG SAMOP 2021

Understanding Variational Quantum Learning Models

JOHANNES JAKOB MEYER, FU BERLIN

@jj_xyz

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The effect of data encoding on the expressive power of variational quantum machine learning models

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Disclaimer: After uploading to the arxiv we were notified of work by Vidal and Theis[1] that has significant overlap with ours.

[1]Francisco Javier Gil Vidal and Dirk Oliver Theis. "Input Redundancy for Parameterized Quantum Circuits." *Frontiers in Physics* 8 (2020): 297.

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FU Berlin

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A lot of work has been done to understand the practical side, but little is known on the theory side

So we asked ourselves: **What functions can such models learn?**

Variational Quantum Learning Models

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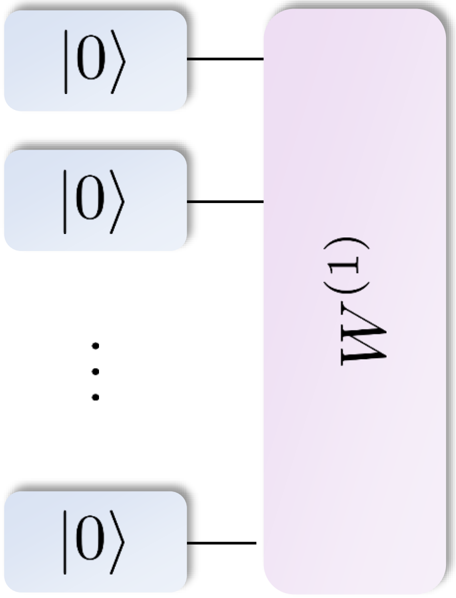
$|0\rangle$

$|0\rangle$

$\vdots$

$|0\rangle$

Variational Quantum Learning Models



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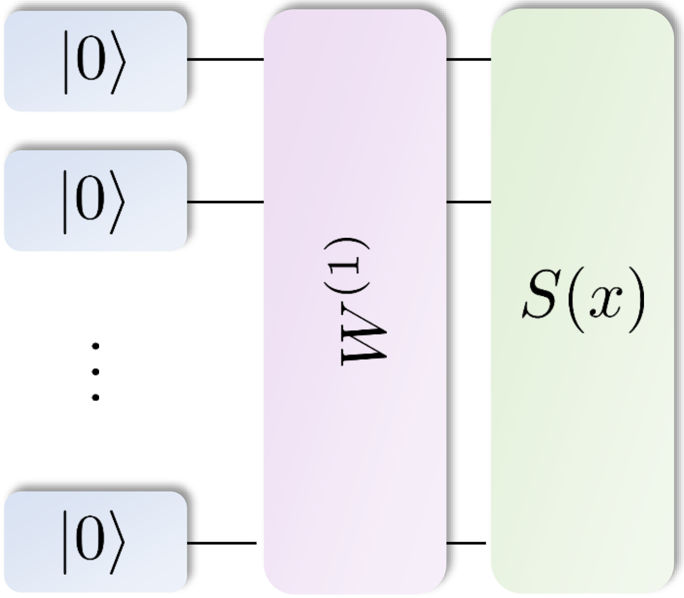

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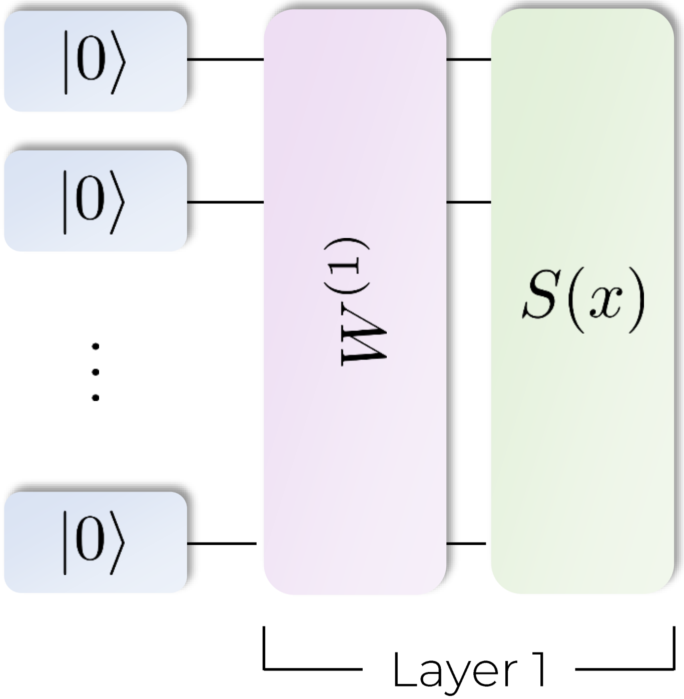

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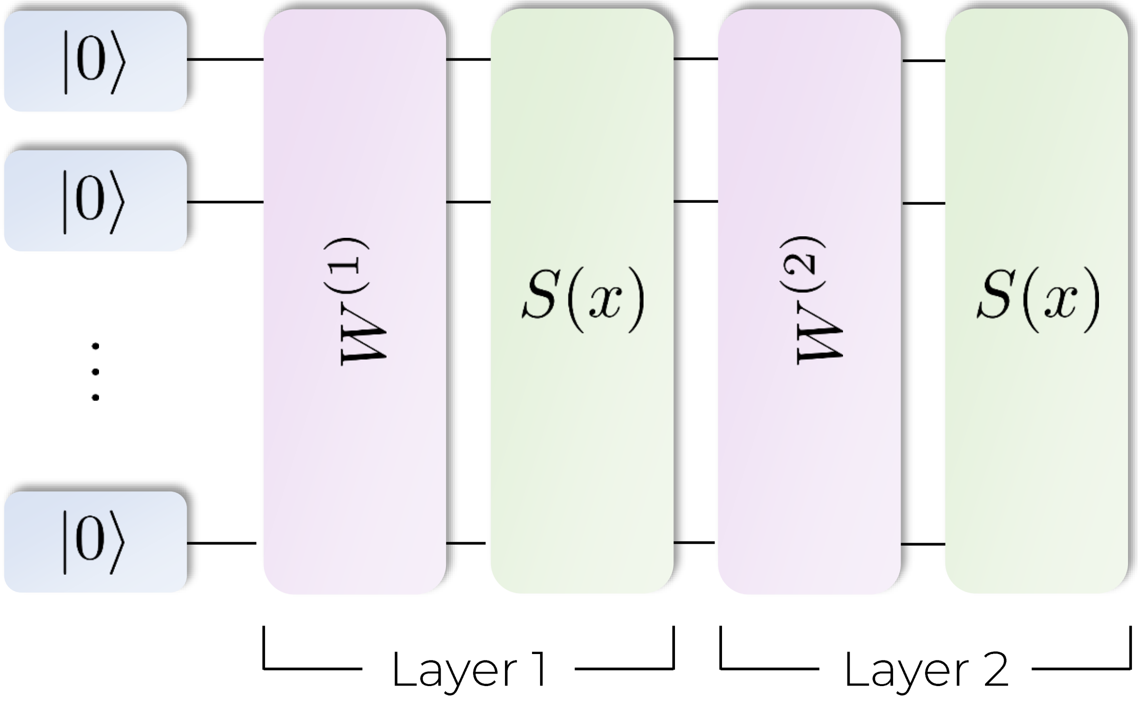

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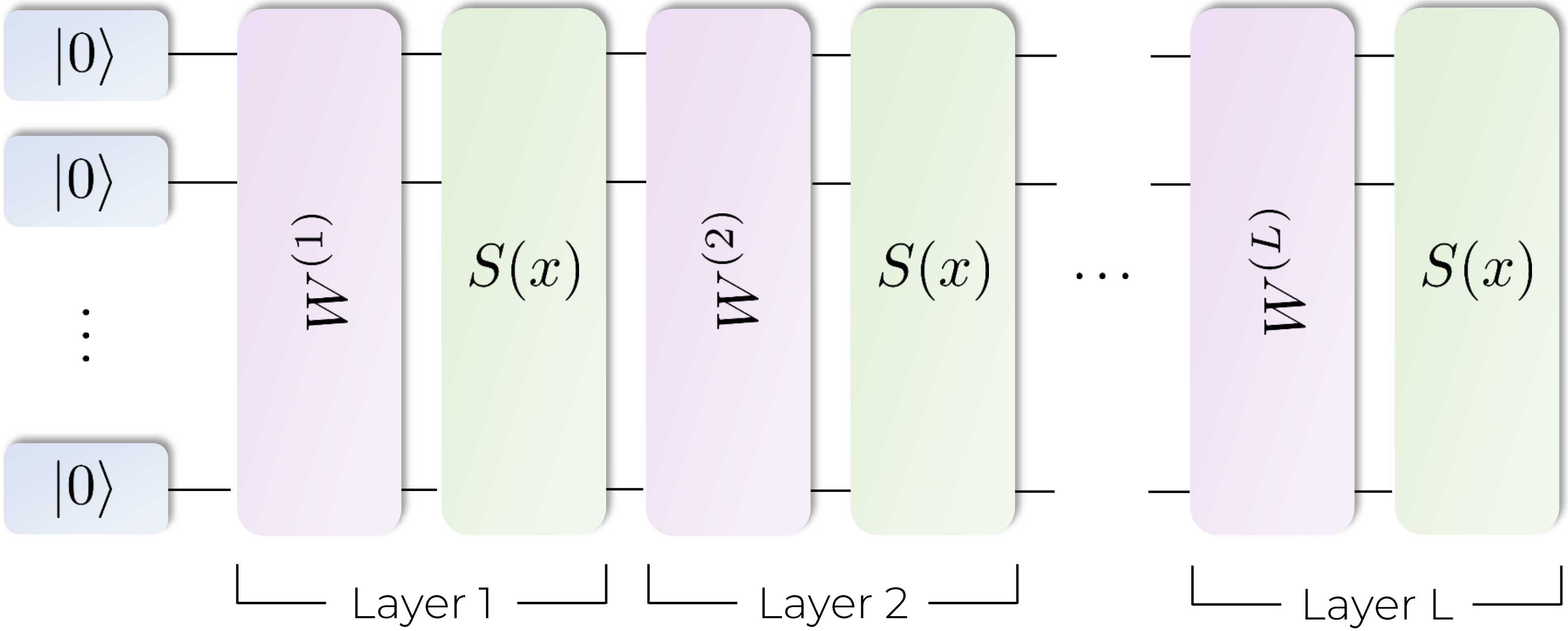

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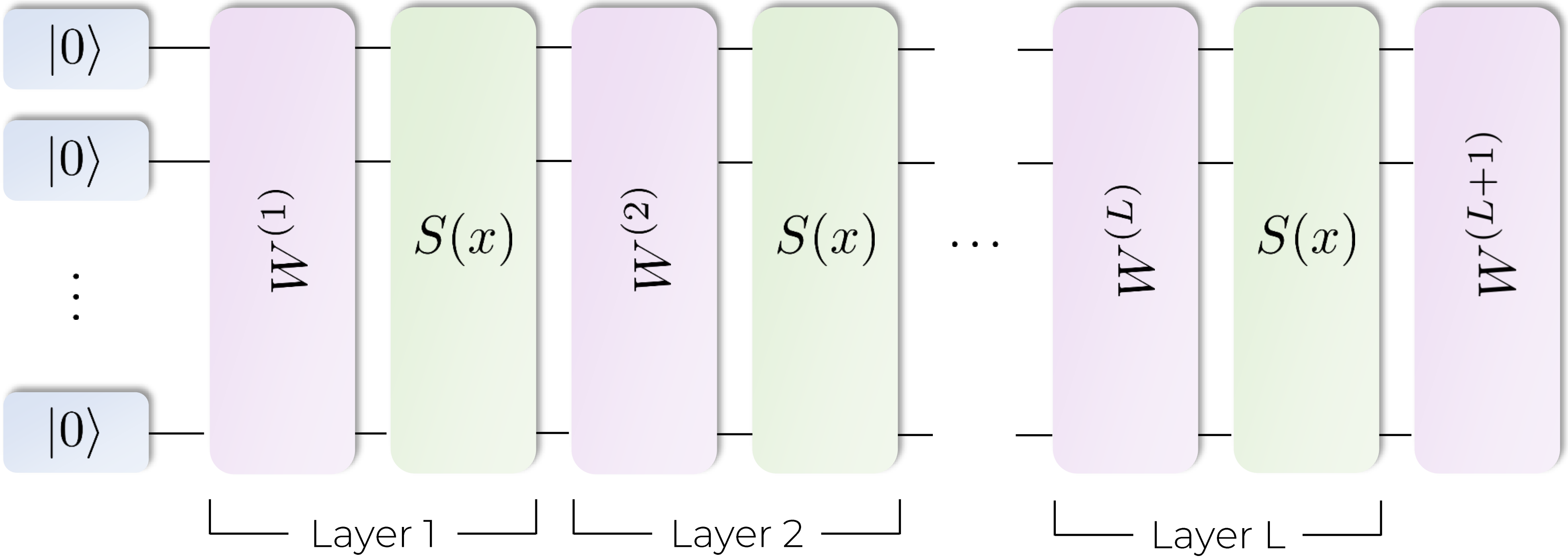

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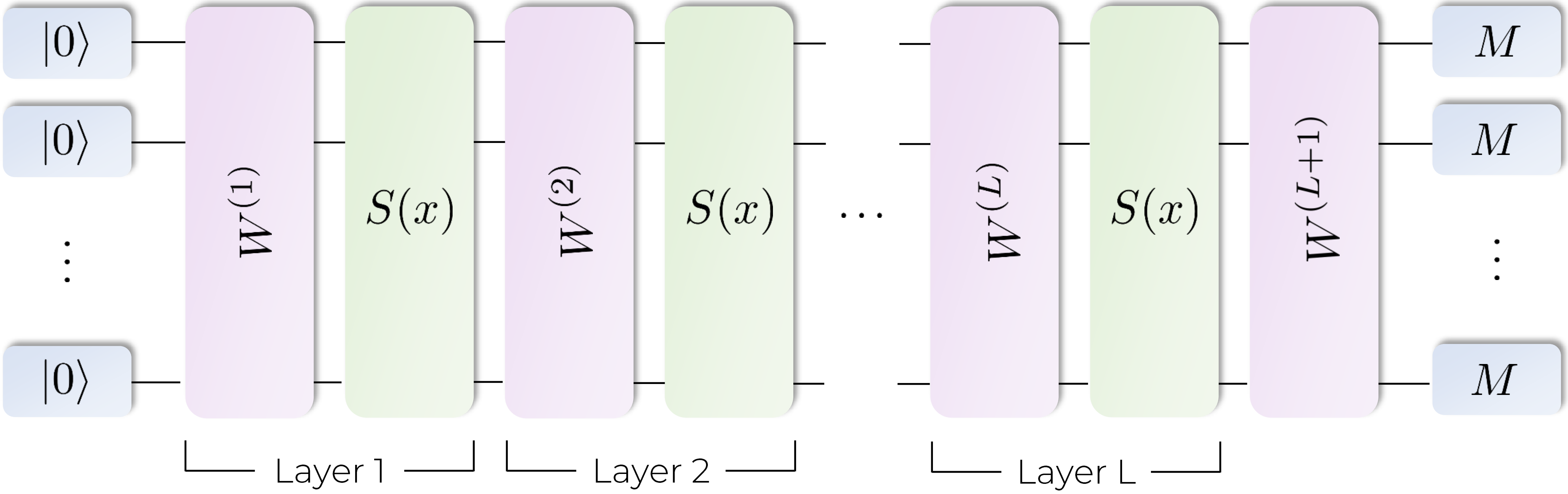

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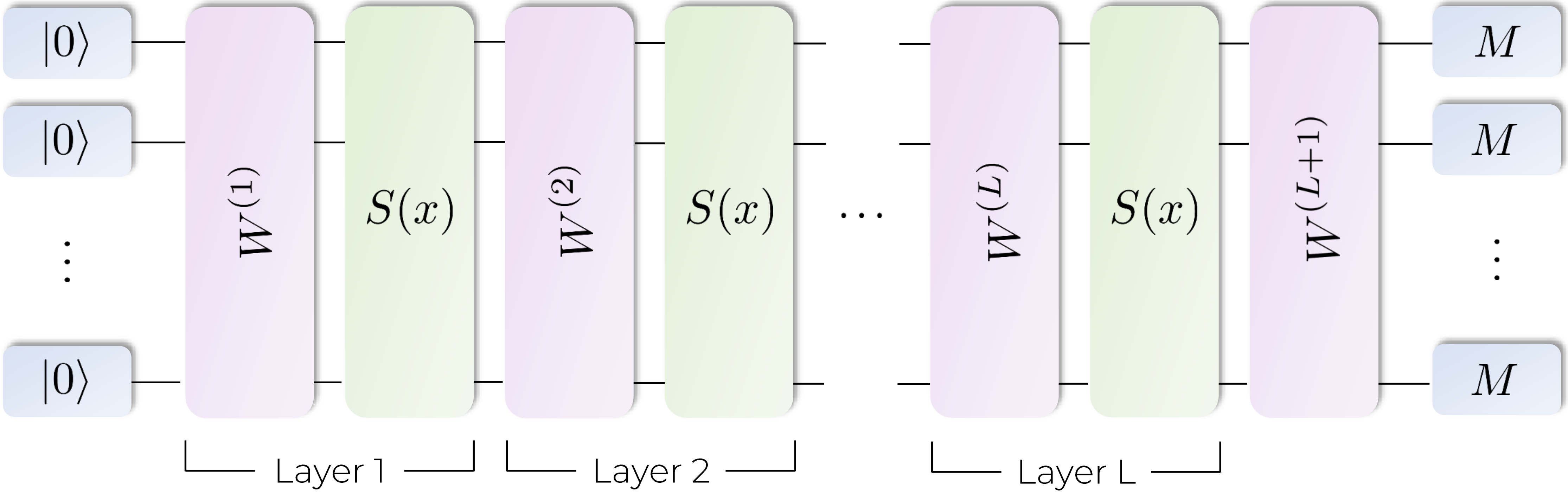
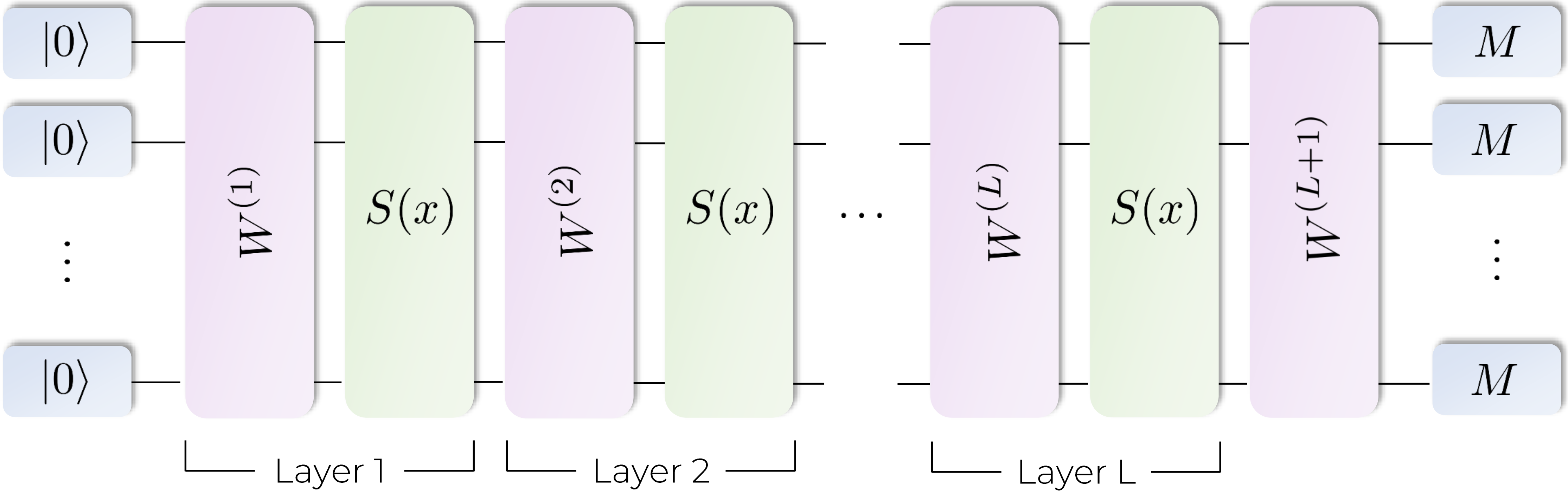


$$f(x) = \langle M \rangle$$

Model Output

Variational Quantum Learning Models



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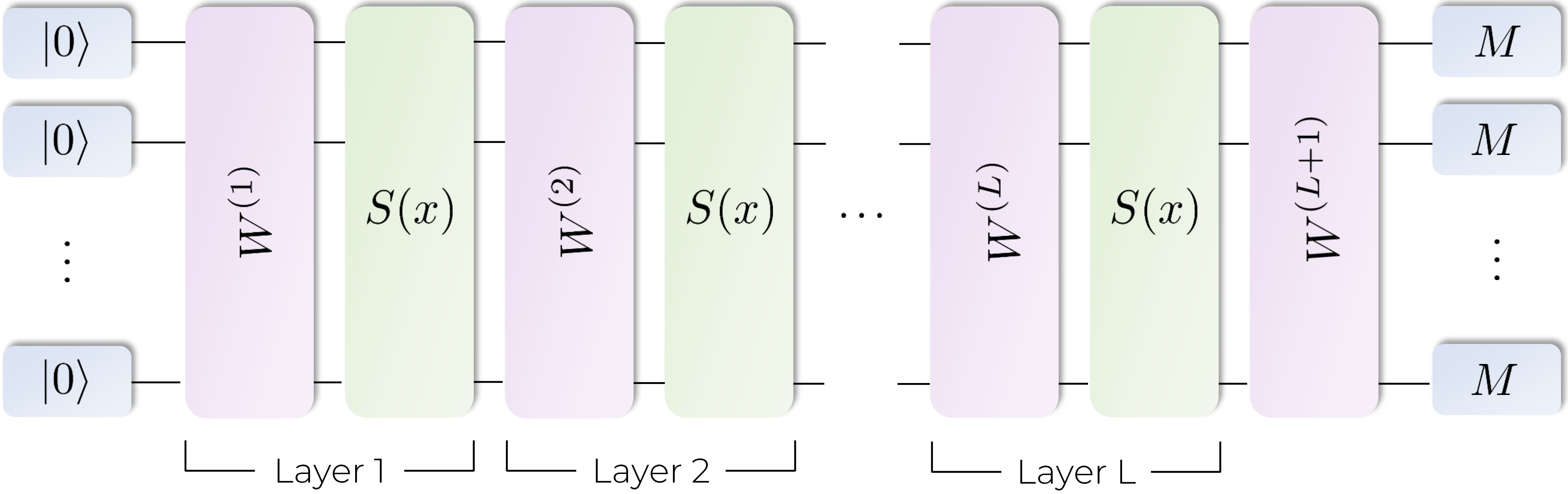
Hamiltonian Evolution

Variational Quantum Learning Models



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Hamiltonian Evolution

$$W^{(l)}(\boldsymbol{\theta})$$

Trainable Blocks

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The output state contains all possible sums of frequencies

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$$f_{M,\boldsymbol{\theta}}(x) = \langle \psi_{\boldsymbol{\theta}}(x) | M | \psi_{\boldsymbol{\theta}}(x) \rangle = \sum_{\omega \in \Omega} c_{\omega}(M, \boldsymbol{\theta}) e^{i\omega x}$$

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For one layer of encoding

$$\Omega = \{\lambda_j - \lambda_k \,|\, \lambda_j, \lambda_k \in \mathrm{spec}(H)\}$$

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The accessible spectrum consists of all **sums of differences** of eigenvalues of the generator of the data encoding

Take-Home Message

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Quantum learning models output Fourier series, repeated data encoding gives access to higher frequencies

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Pauli rotations are the most popular encoding strategy, e.g.

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The number of available frequencies grows linearly in depth and width

But for general encodings the dependence can be exponential:

$$\#(\text{frequencies}) \leq \frac{d^{2L}}{2} - 1$$

d System dimension

L Number of layers

Consequences for Learning Tasks

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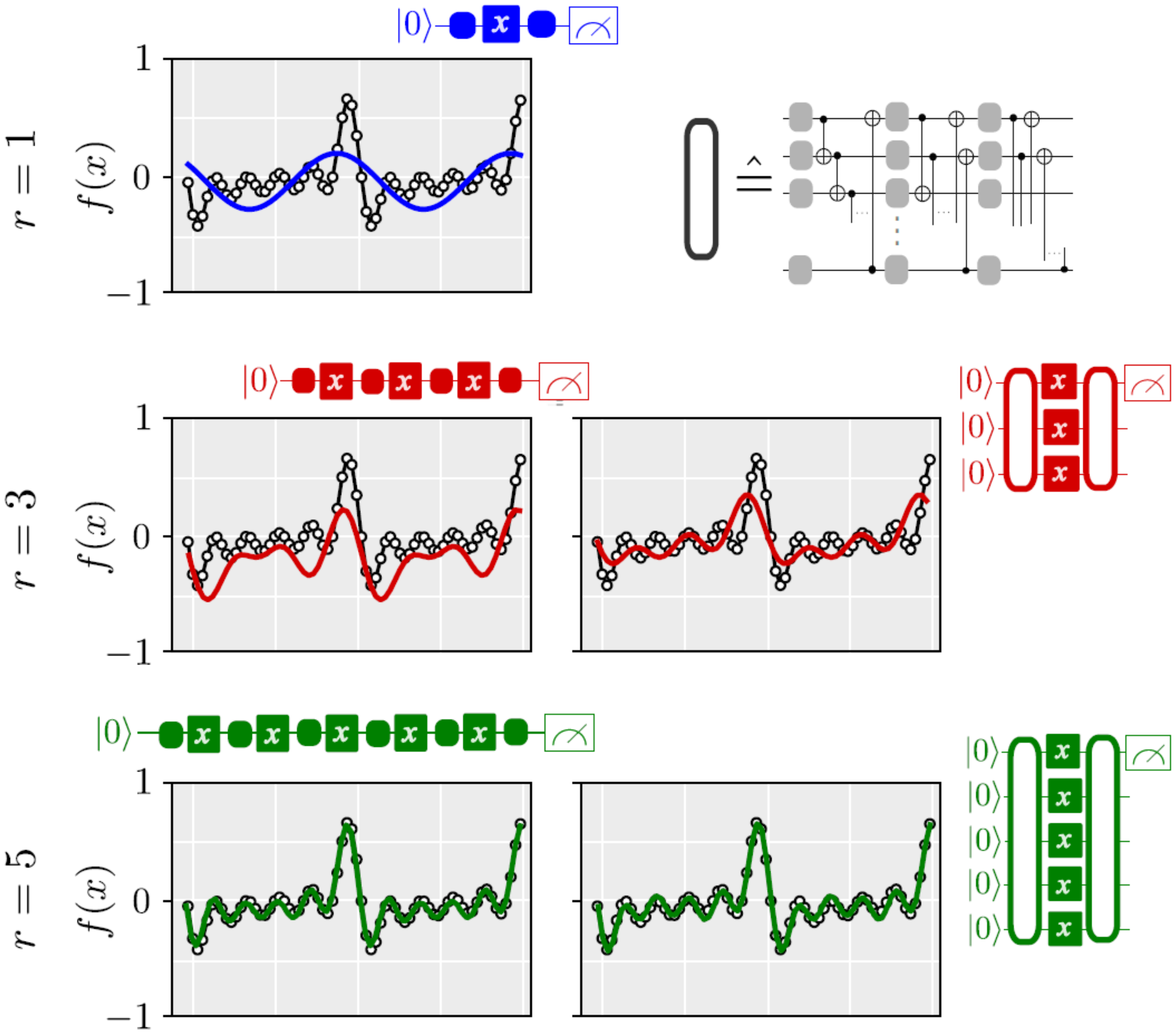

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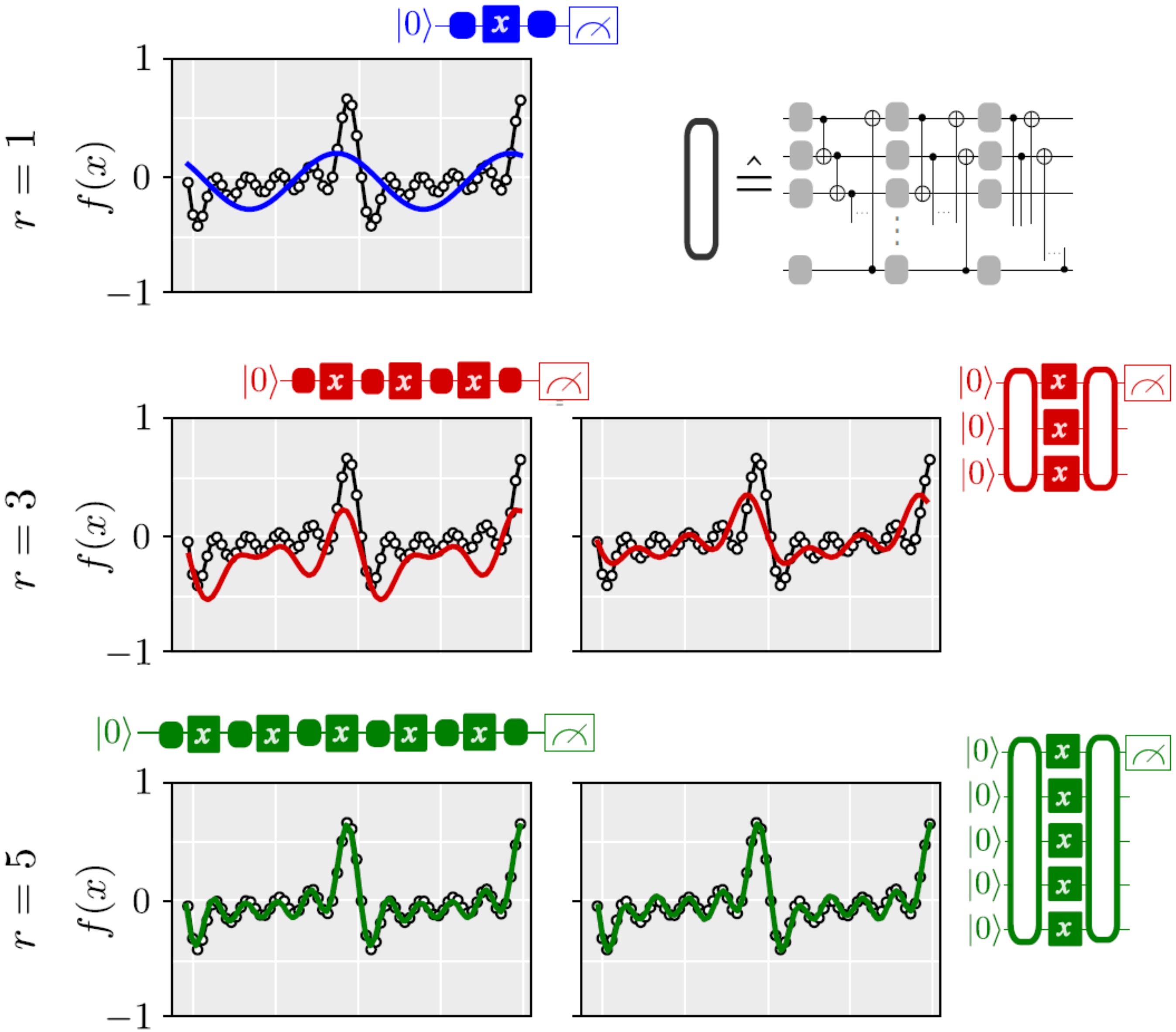

You can reproduce all figures from the paper at home!

arXiv:2106.03880

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Encoding-dependent generalization bounds for parametrized quantum circuits

Matthias C. Caro,[1,2] Elies Gil-Fuster,[3] Johannes Jakob Meyer,[3,4] Jens Eisert,[3,5] and Ryan Sweke[3]

[1]*Department of Mathematics, Technical University of Munich, 85748 Garching, Germany*
[2]*Munich Center for Quantum Science and Technology (MCQST), 80799 Munich, Germany*
[3]*Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany*
[4]*QMATH, Department of Mathematical Sciences, University of Copenhagen, 2100 Copenhagen, Denmark*
[5]*Helmholtz-Zentrum Berlin für Materialien und Energie, 14109 Berlin, Germany*

(Dated: June 9, 2021)

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Matthias Caro
TU Munich

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We want to quantify how good a trained learning model performs

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Loss

Measures the quality of the learning model‘s predictions

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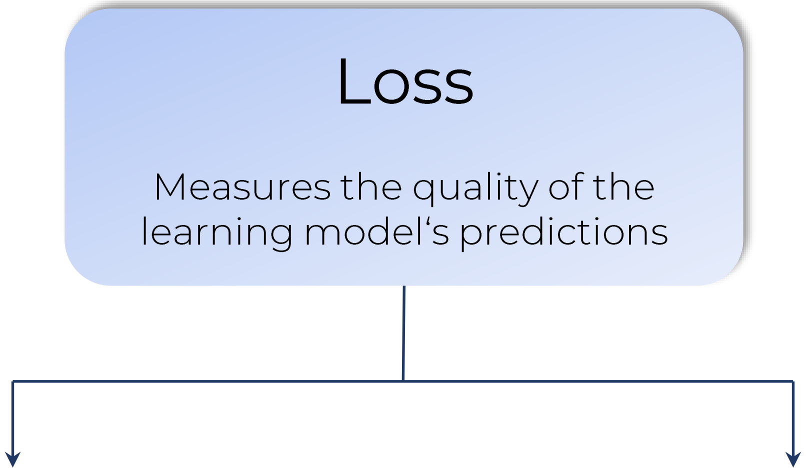

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Model loss averaged over the *data distribution*

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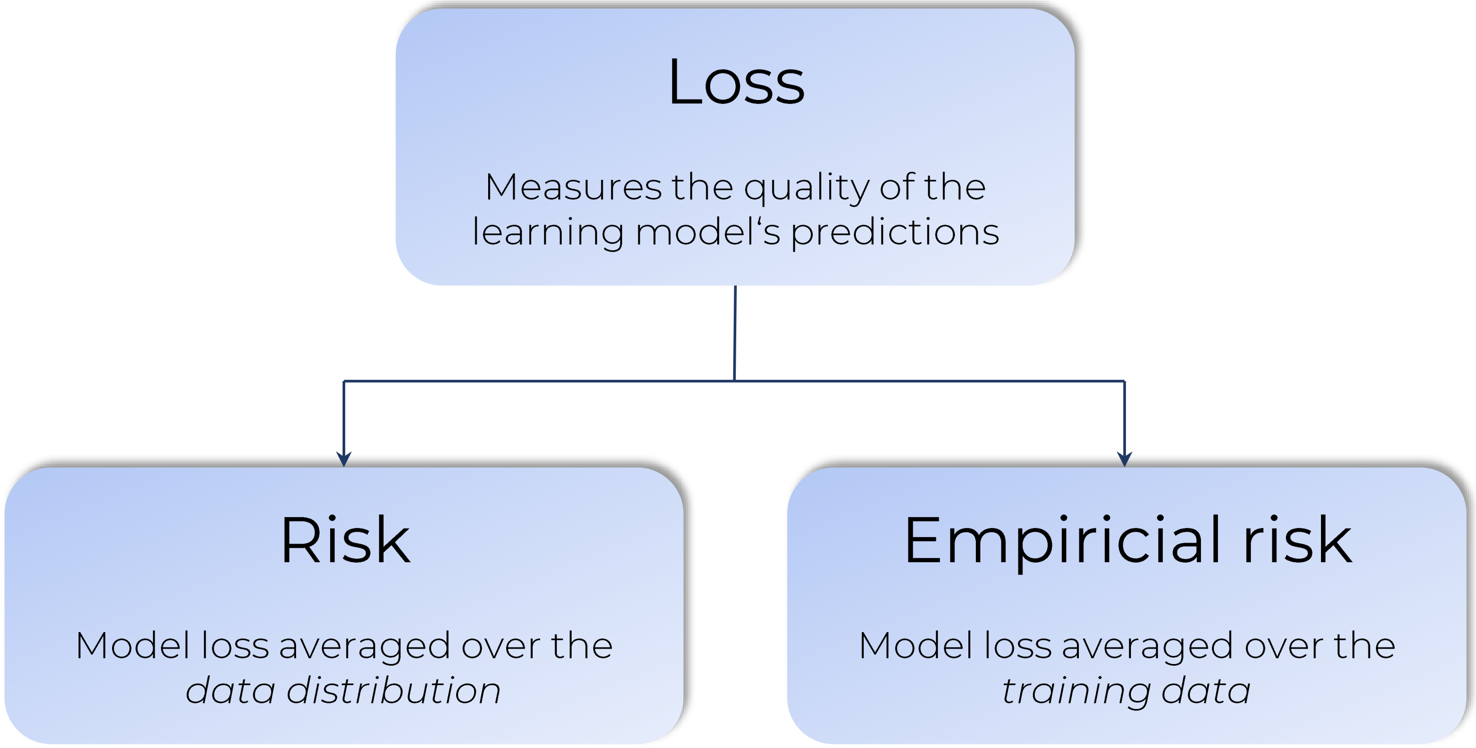

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Empiricial risk

Model loss averaged over the *training data*

Generalization Bounds

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Measures the quality of the learning model's predictions

Risk		**Empiricial risk**			**Generalization bound**
Model loss averaged over the *data distribution*	$-$	Model loss averaged over the *training data*	$\vert$	$\leq$	Holds with high probability

Deriving a Generalization Bound

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Generalization bounds reflect the „complexity" of the model

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How many points do I need such that any possible output of the learning model is ϵ-close to at least one point?

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We exploit the Fourier representation to derive covering numbers that depend explicitly on the data-encoding strategy

Results

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Models that use fixed Hamiltonians have a generalization bound that scales polynomially in the number of gates, guaranteeing efficient learning

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Models using arbitrary Hamiltonians can have a generalization bound that scales exponentially

Take-Home Message

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Quantum learning models with fixed Hamiltonians can learn efficiently

Thank you for your attention!

Paper Data-encoding and Fourier series	Paper Generalization bounds	Slides	Demo